Quantum phase transition with dissipative frustration

arXiv:1711.11346

D. Maile, S. Andergassen, W. Belzig, G. Rastelli

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Zukunftskolleg
free • creative • connecting

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SFB 767

EBERHARD KARLS
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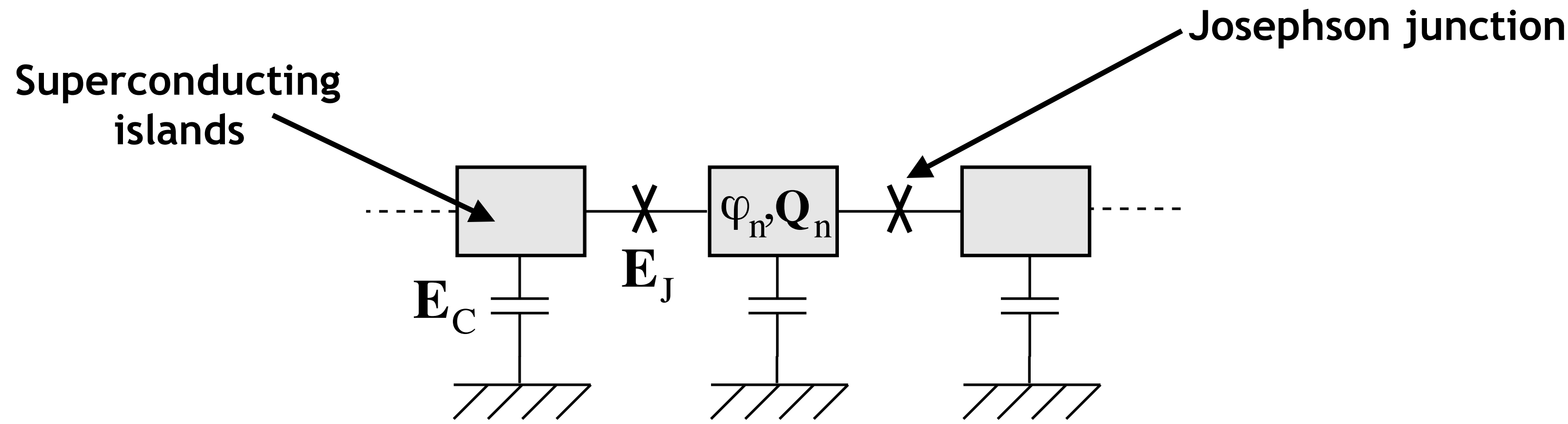
Chain of Josephson junctions



Mesoscopic quantum many-body system:

- Engineered interactions and tunable parameters
- Driving, state preparation and nonequilibrium dynamics

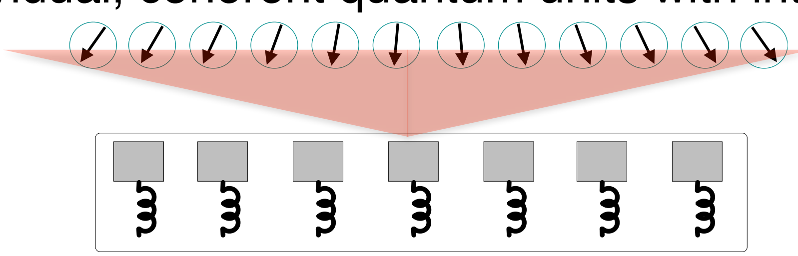
Building block for future quantum applications!

"macroscopic size"

↓

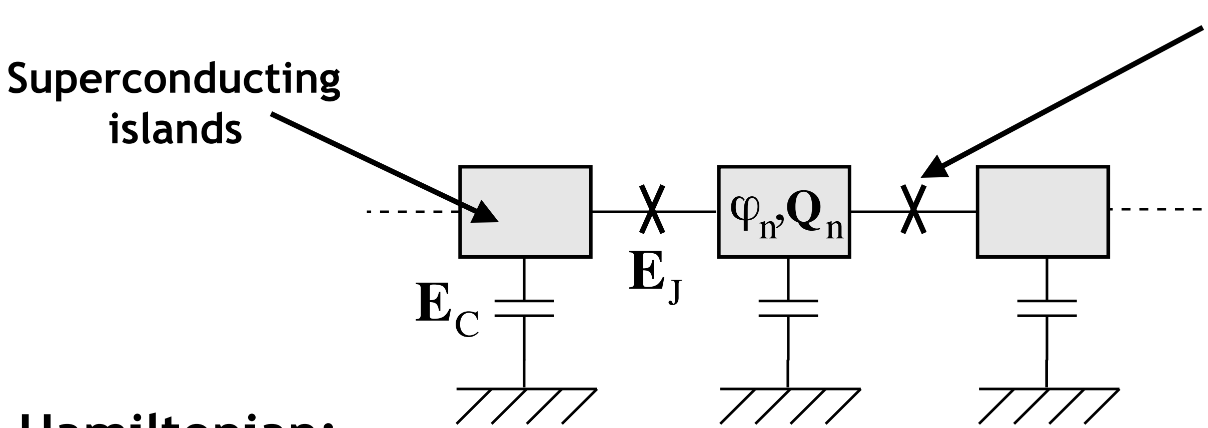
open (dissipative) systems

(individual, coherent quantum units with interaction)



environment

Chain of Josephson junctions



$\Delta \longrightarrow$ large

no QP excitations

Model Hamiltonian:

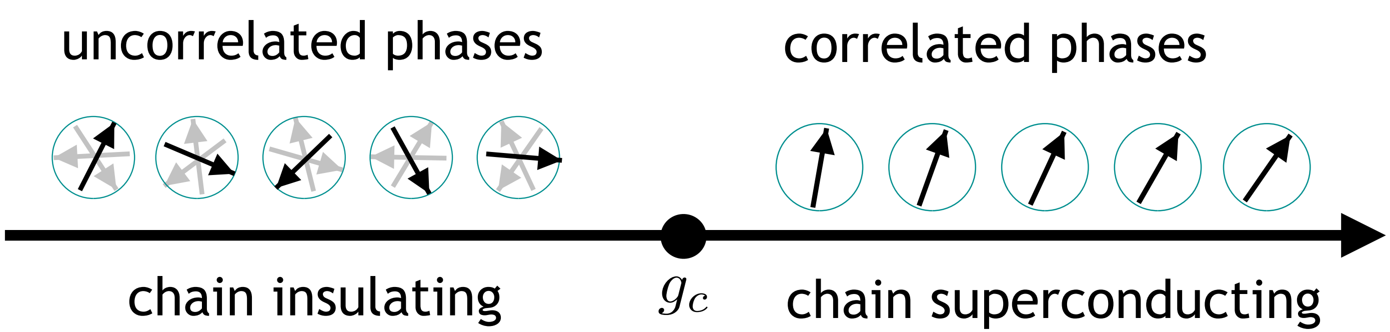
$$\hat{H} = \sum_n \left[\frac{\hat{Q}_n^2}{2C_0} - E_J \cos\left(\Delta\hat{\varphi}_n\right) \right]$$

$$\left[\hat{\varphi}_n, \hat{Q}_m\right] = 2e\, i\, \delta_{nm}$$

$$\Delta\hat{\varphi}_n = \hat{\varphi}_n - \hat{\varphi}_{n-1}$$

$$E_C = 4e^2/C_0$$

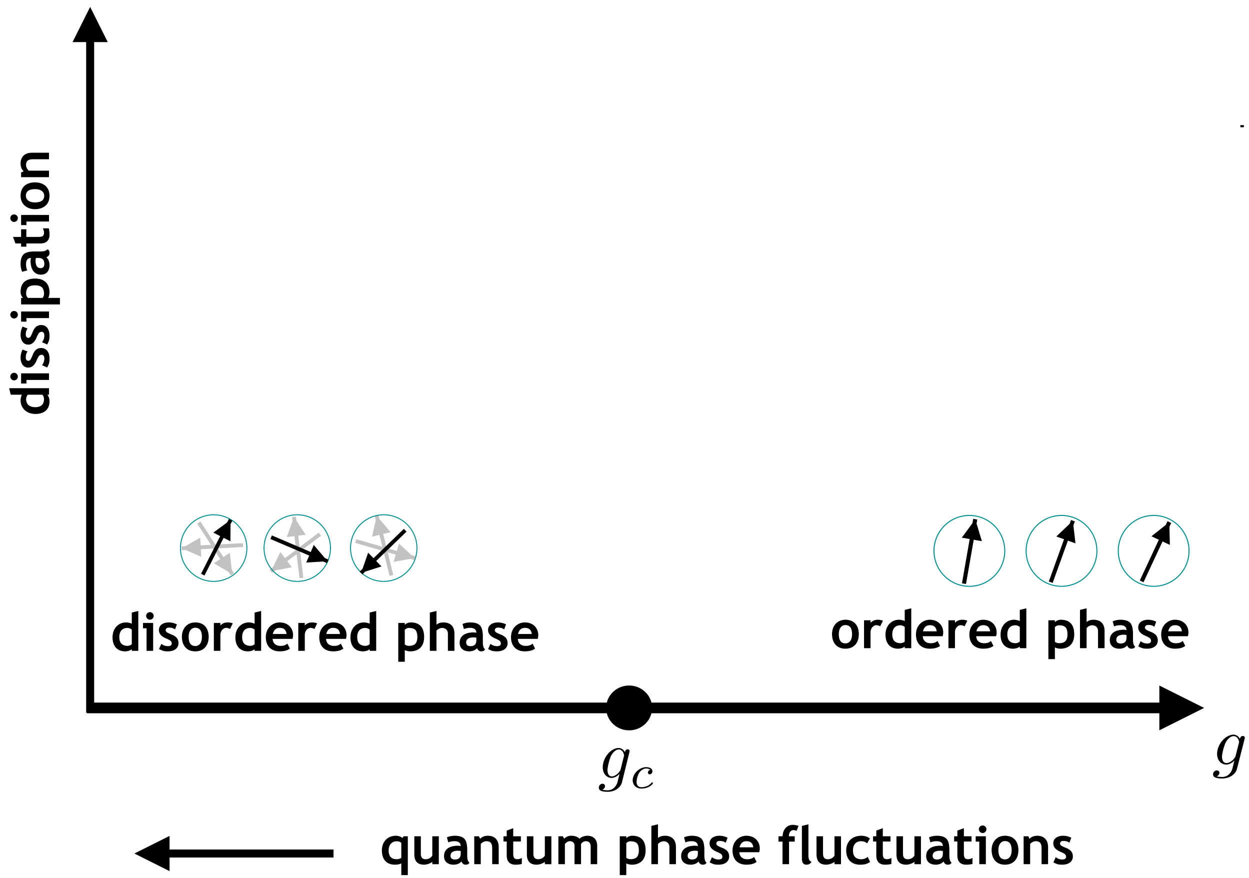
Quantum phase transition $T = 0K$

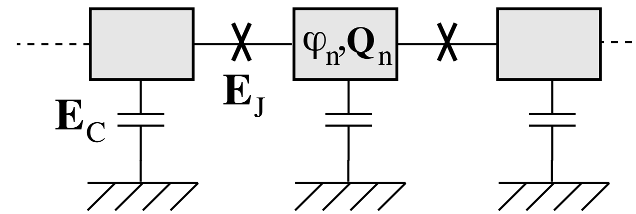


$$g = \sqrt{\frac{E_J}{E_C}}$$

[Bradley, Doniach Phys. Rev. B **30**, 1138 (1984)]

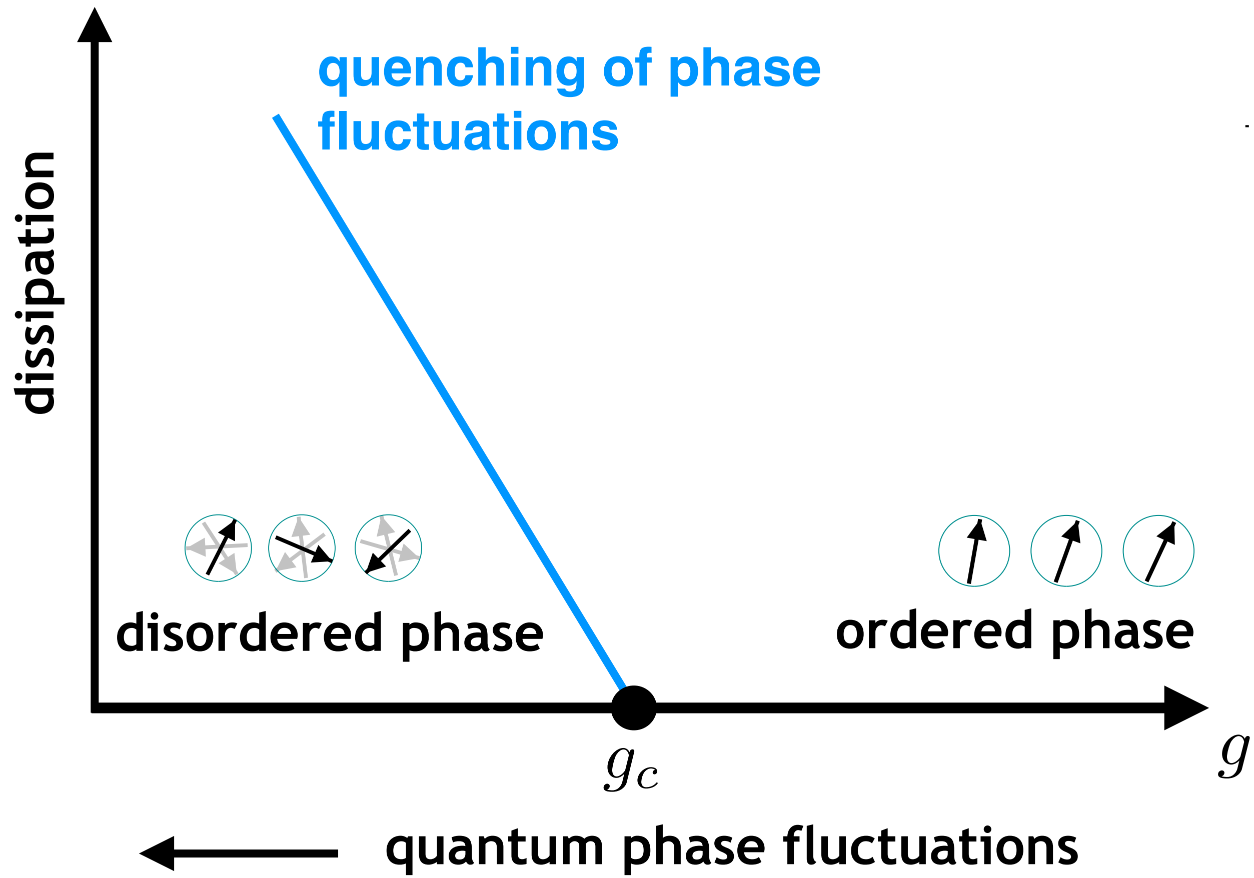
Dissipative quantum phase transition

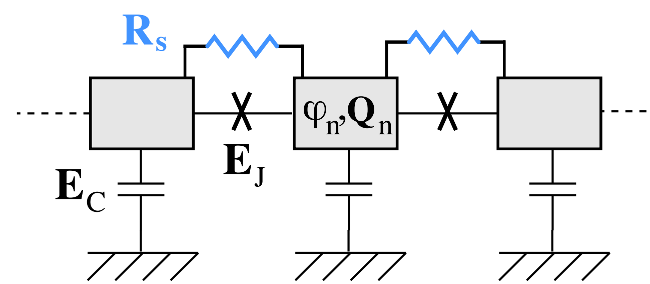




Dissipative quantum phase transition

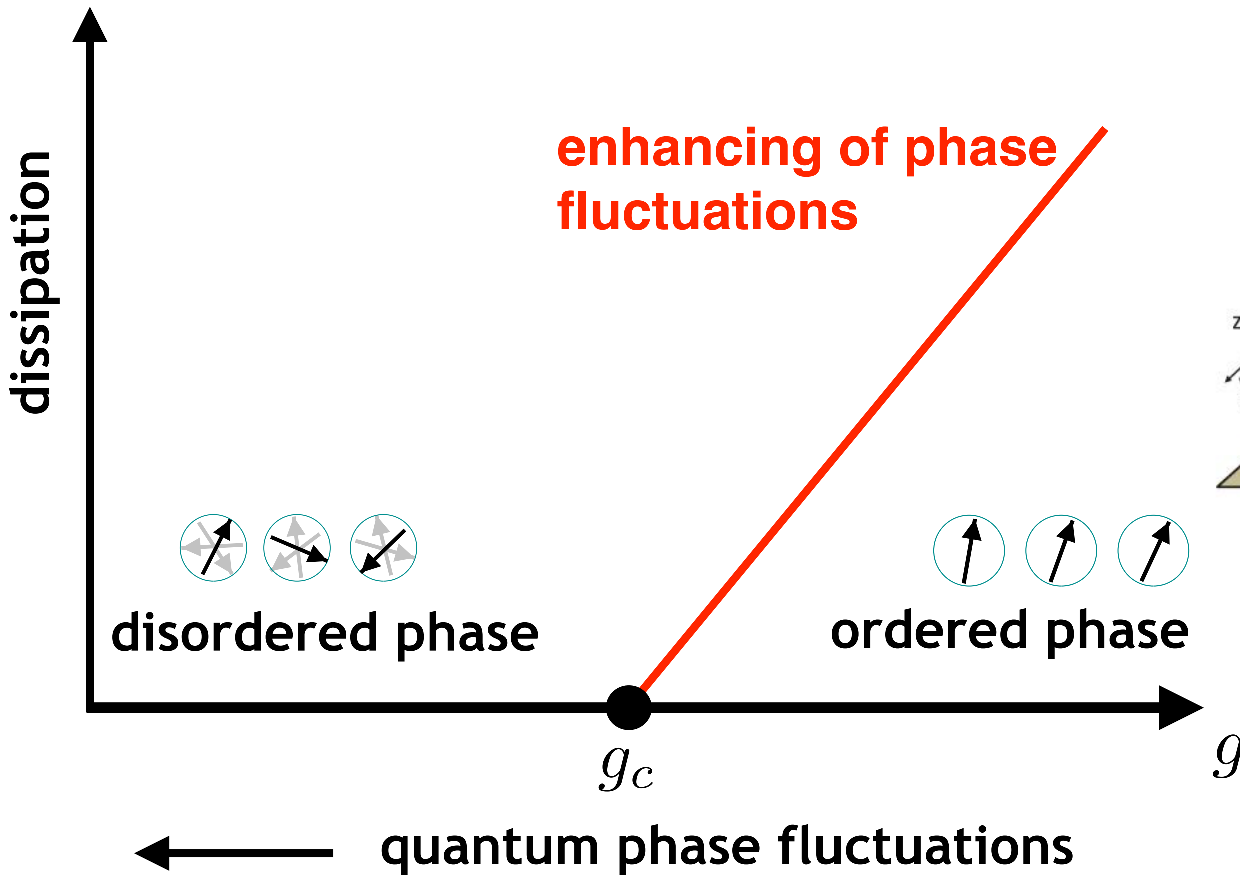
[Bobbert, Fazio, Schön, Zaikin, Phys. Rev. B **45**, 2294 (1992)]

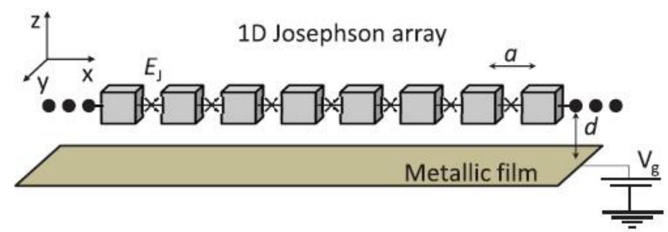




Dissipative quantum phase transition

[Lobos, Giamarchi, Phys. Rev. B **84**, 024523 (2011)]

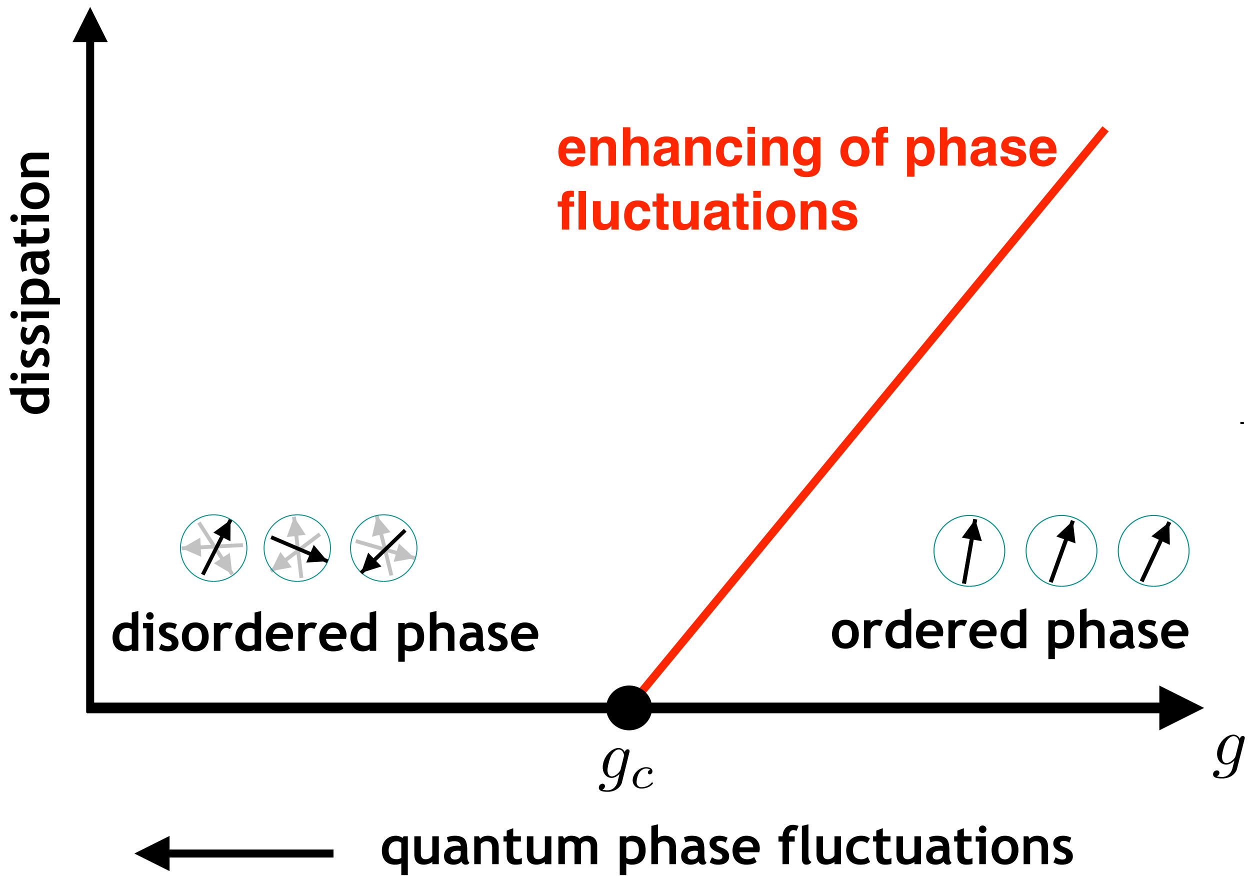


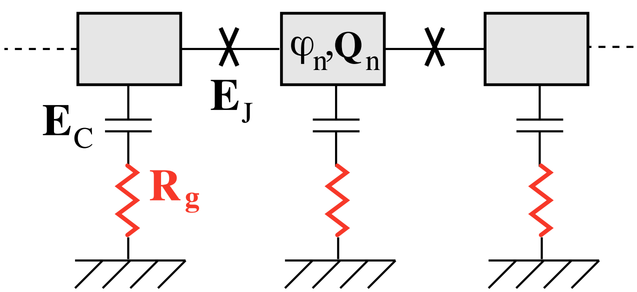


$$\langle \varphi_n^2 \rangle \langle Q_n^2 \rangle \geq e^2$$

Dissipative quantum phase transition

[Maile, Andergassen, Belzig, Rastelli, arXiv:1711.11346 (2017)]





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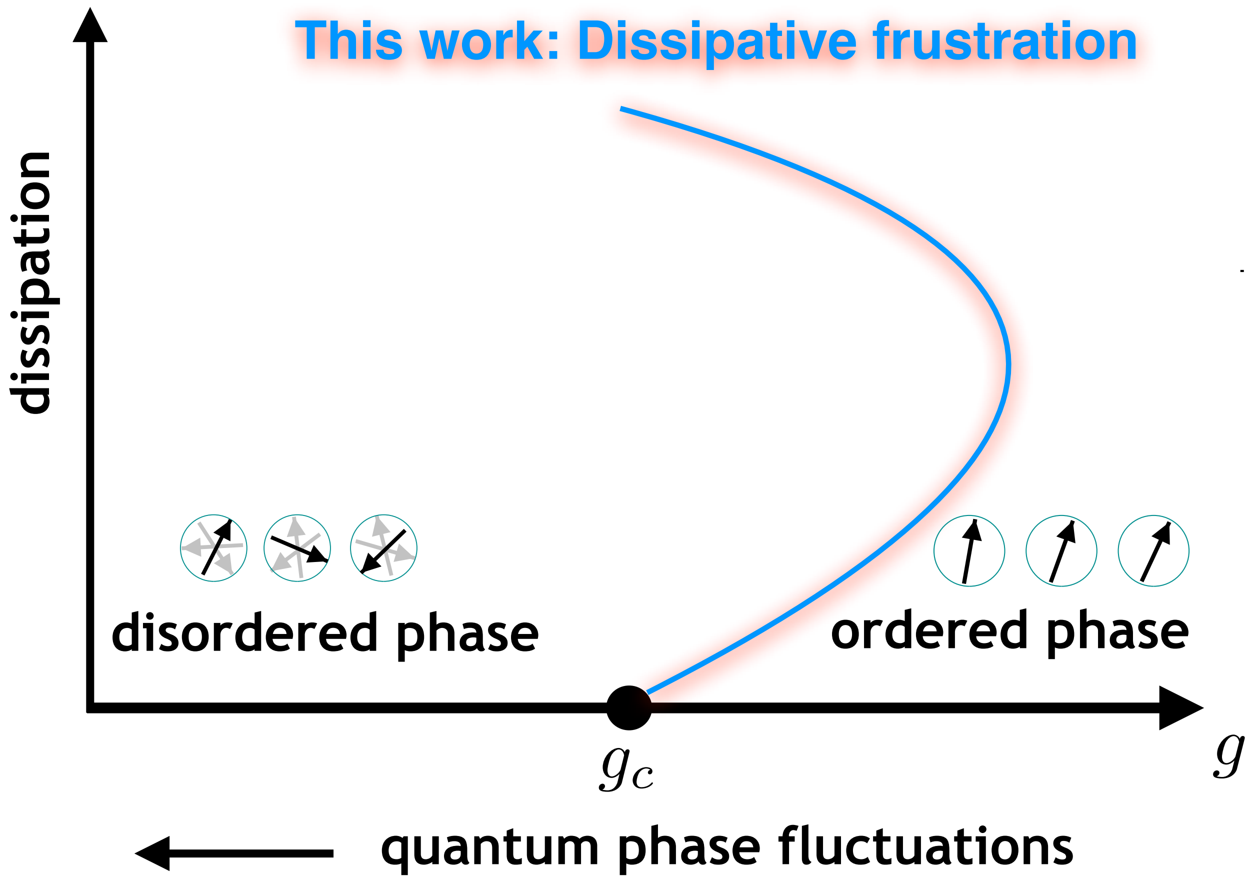

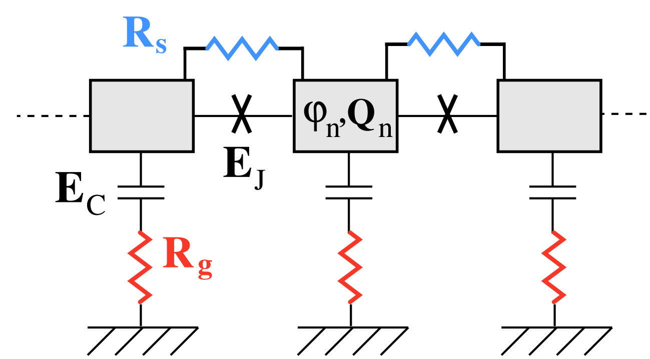

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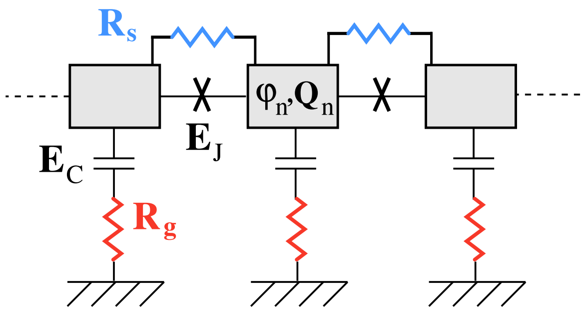
Method and Approximations

Path integral: Euclidean action

$$S_{eff} = \sum_n \left\{ -\int_0^\beta d\tau E_J \cos\left(\Delta\varphi_n(\tau)\right) + \frac{1}{2} \iint_0^\beta d\tau d\tau' \left[K(\tau-\tau') \left|\Delta\varphi_n(\tau) - \Delta\varphi_n(\tau')\right|^2 + \widetilde{K}(\tau-\tau')\dot{\varphi}_n(\tau)\dot{\varphi}_n(\tau') \right] \right\}$$



$$R_q = \frac{h}{2e^2}$$

$$\alpha = \frac{R_q}{R_s}$$

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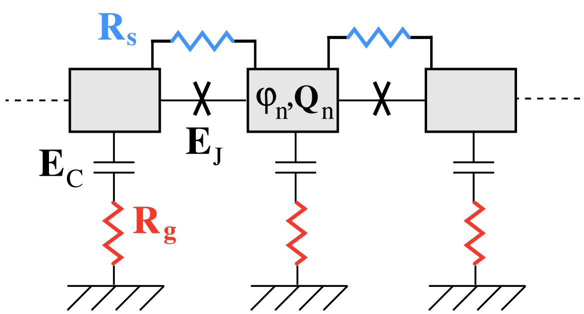

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Self-consistent harmonic approximation

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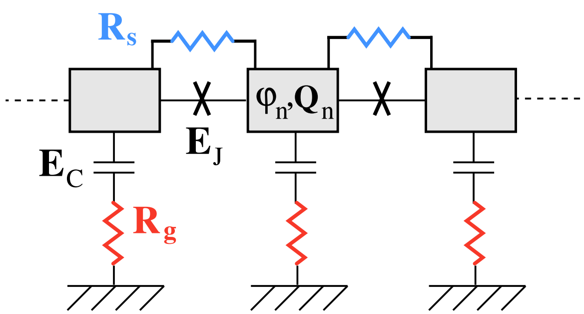

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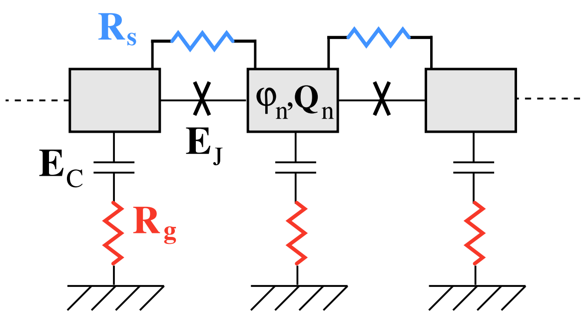

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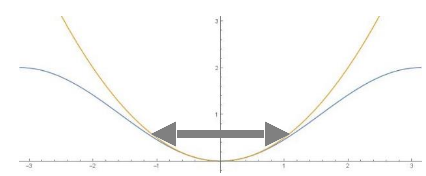

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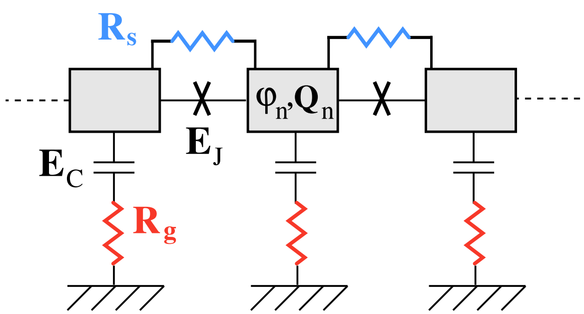

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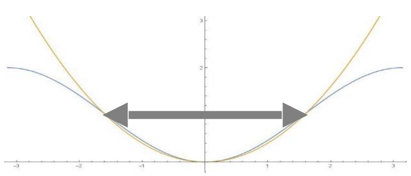

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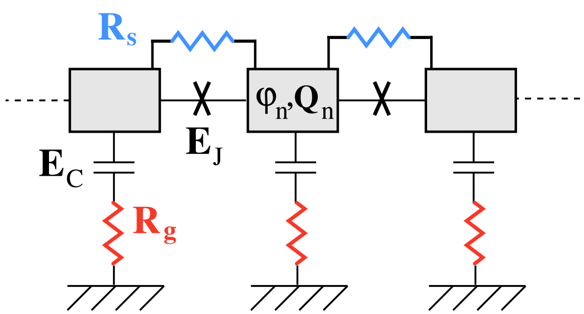

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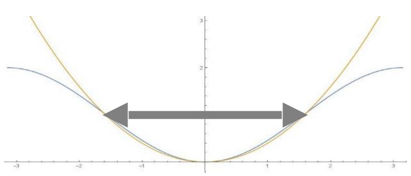

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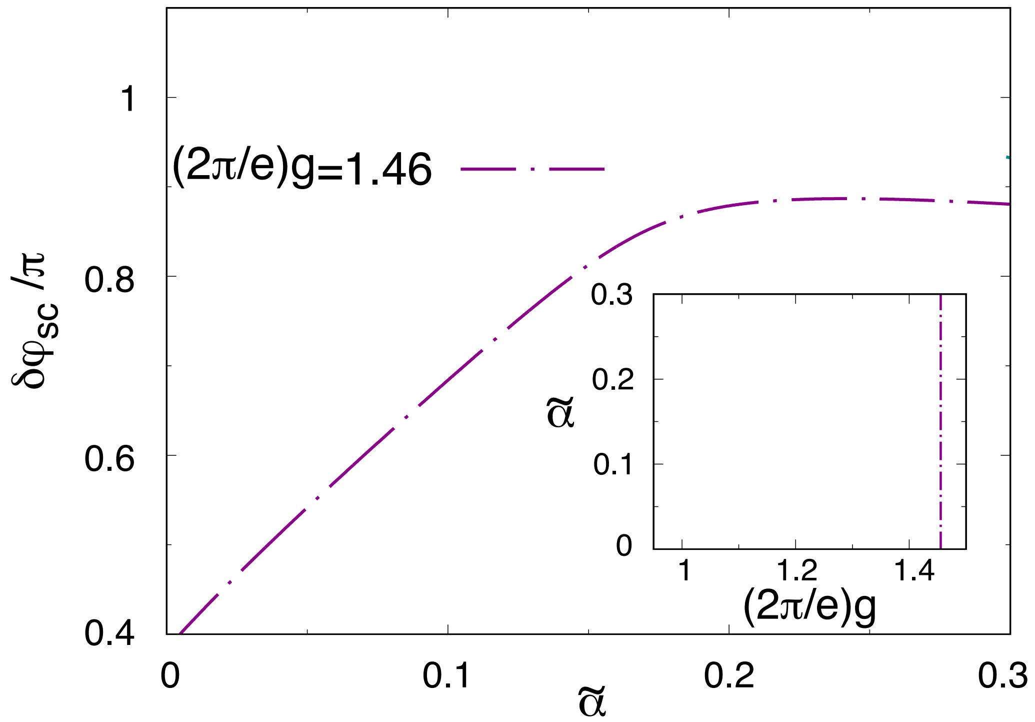
$$\delta\varphi_{sc}(g, \alpha, \widetilde{\alpha}, V_{sc}) = \sqrt{\langle \Delta\varphi^2 \rangle_{sc}}$$

Zero temperature fluctuations of the phase difference

Example: solution of the self-consistent equation

$$\widetilde{\alpha}/\alpha = 0.3$$



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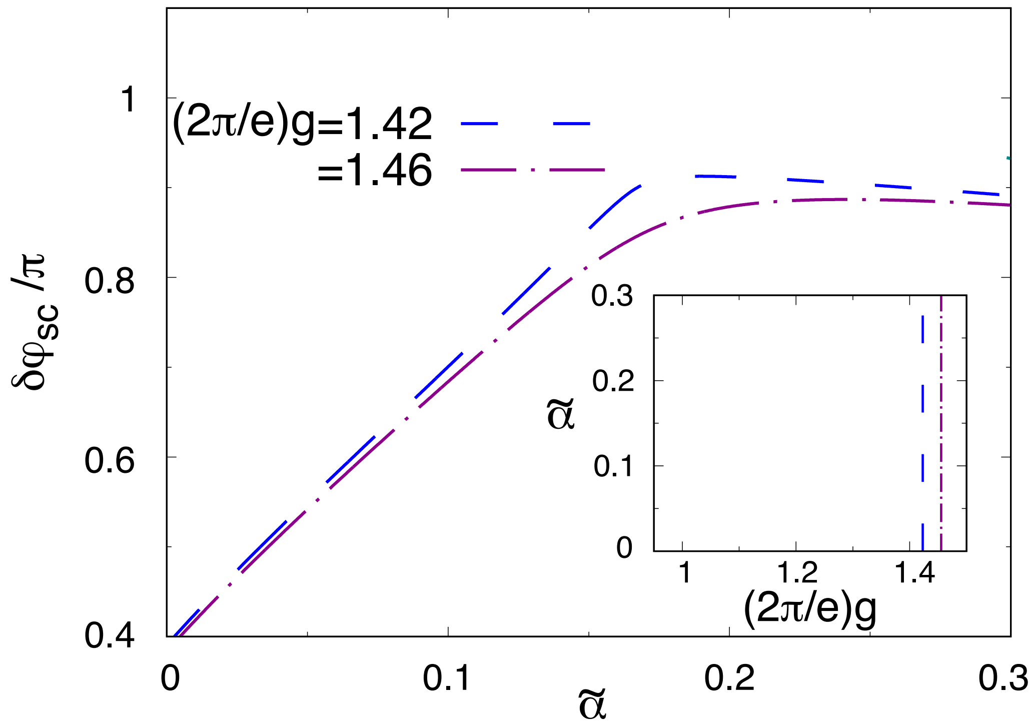

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non-monotonic fluctuations

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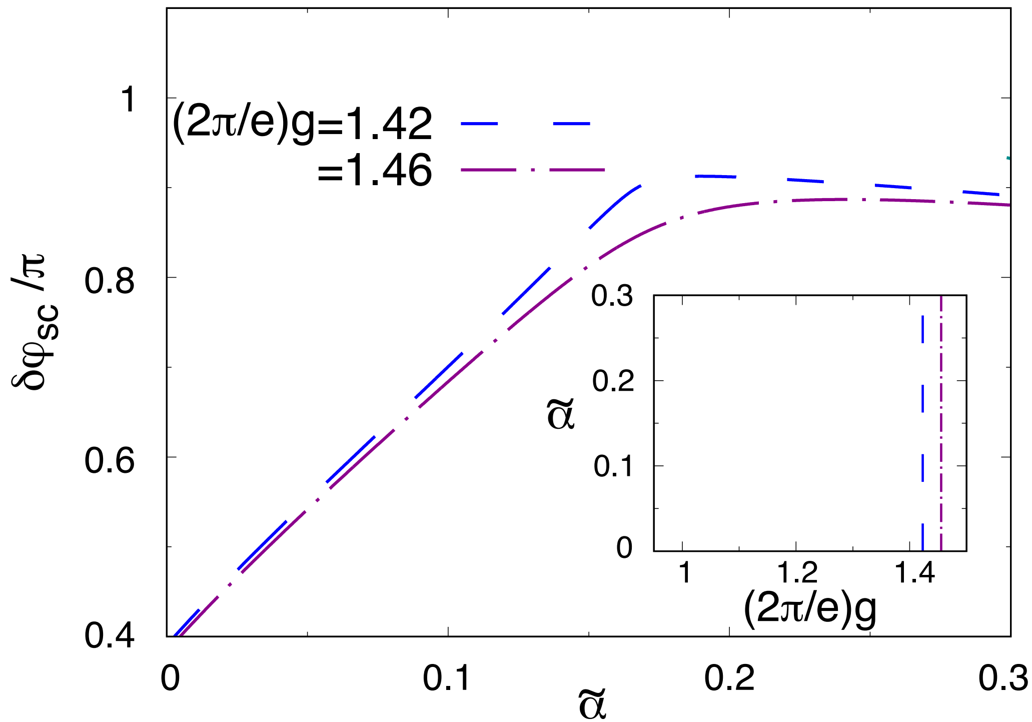

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Dissipative frustrated harmonic oscillator

[G. Rastelli, New J. Phys. **18**, 053033 (2016)]

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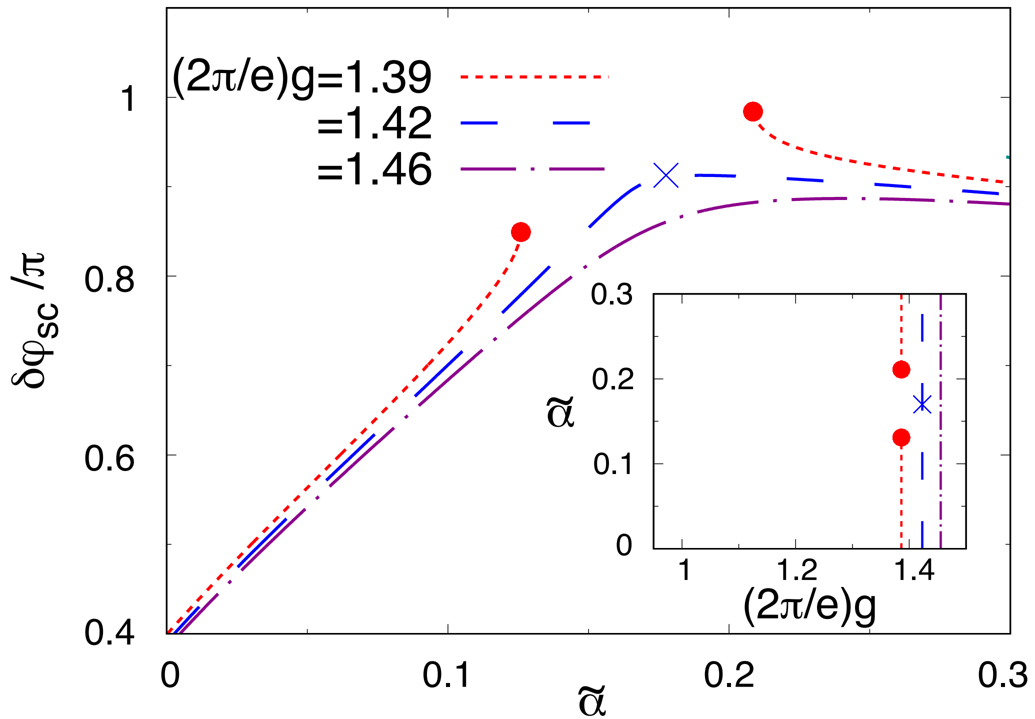

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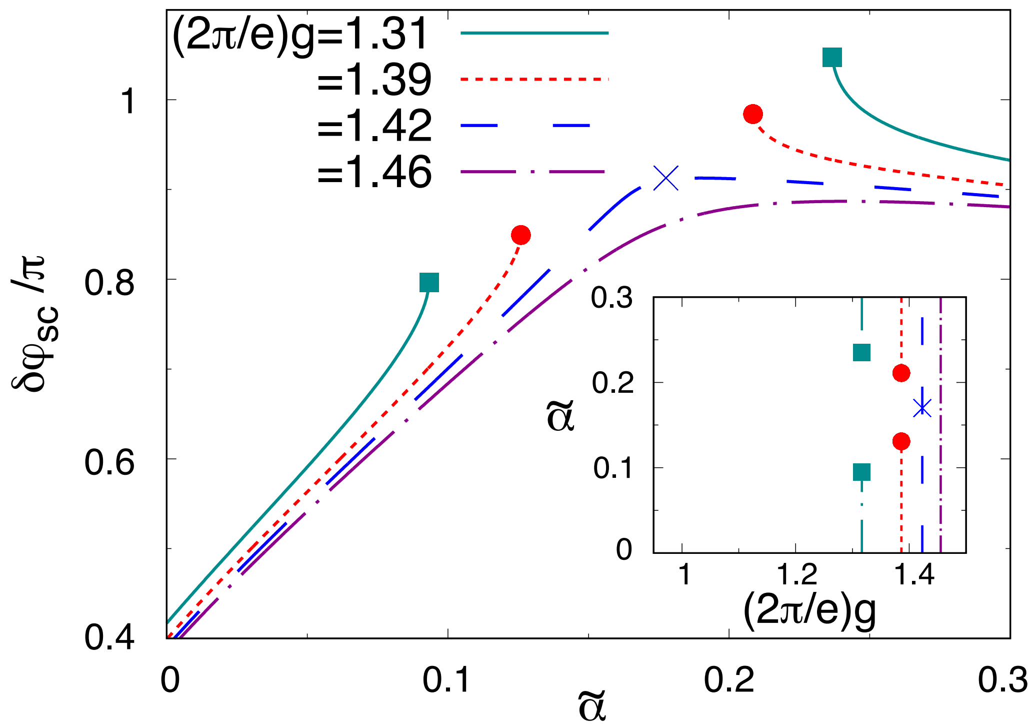

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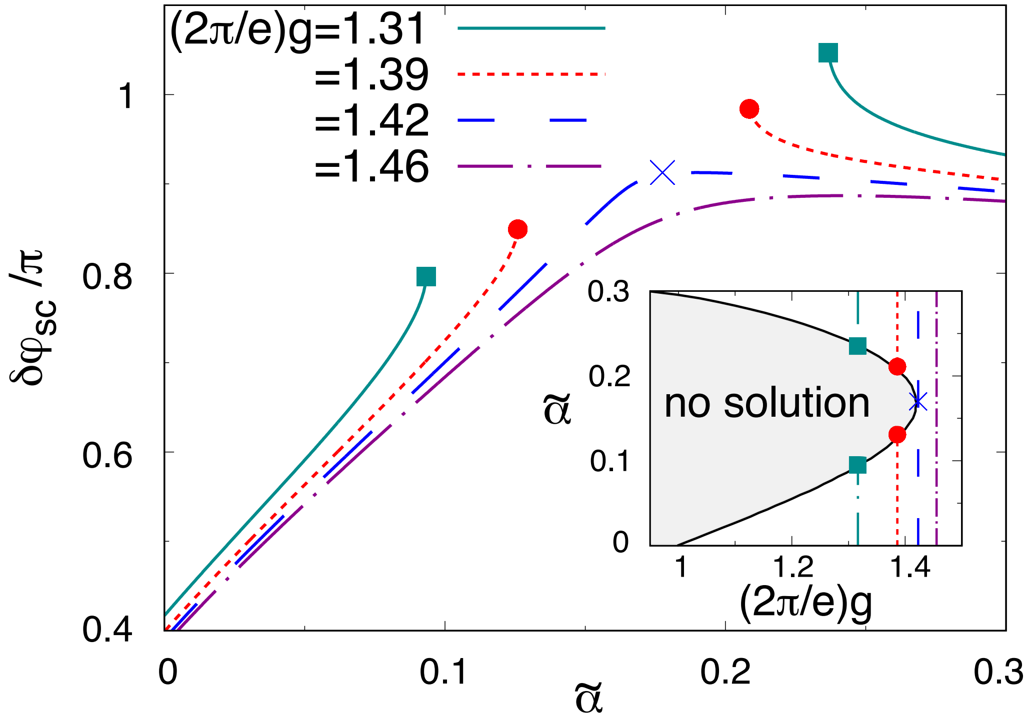


non-monotonic fluctuations → non-monotonic critical line

↓

Dissipative frustrated harmonic oscillator

[G. Rastelli, New J. Phys. **18**, 053033 (2016)]

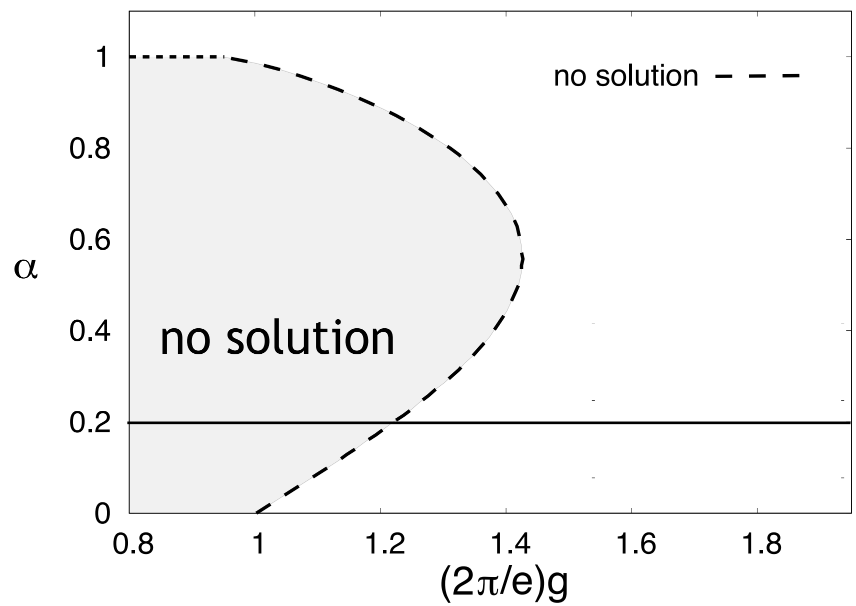
Phase diagram $\widetilde{\alpha}/\alpha = 0.3$

Critical solution of self-consistent equation:

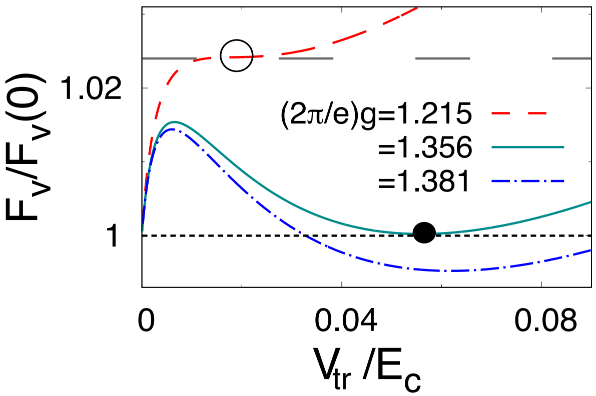
○ $$V_{sc} = E_J e^{-\frac{1}{2}\langle \Delta\varphi^2 \rangle_{sc}}$$

Condition for phase transition in variational approach:

● $$F_v(0) = F_v(V_{sc})$$



$\alpha = 0.2$



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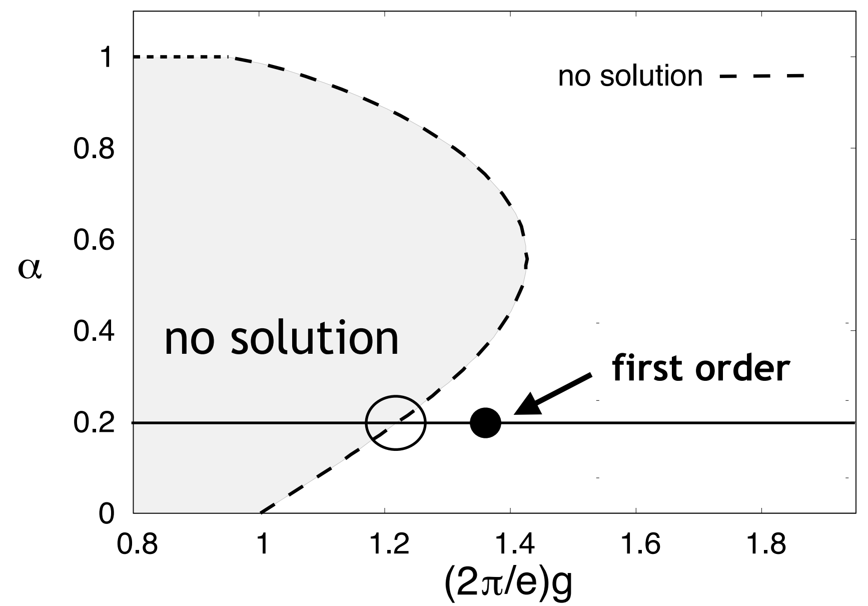

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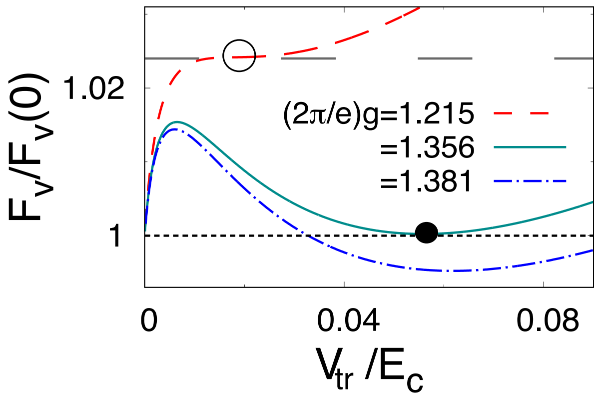

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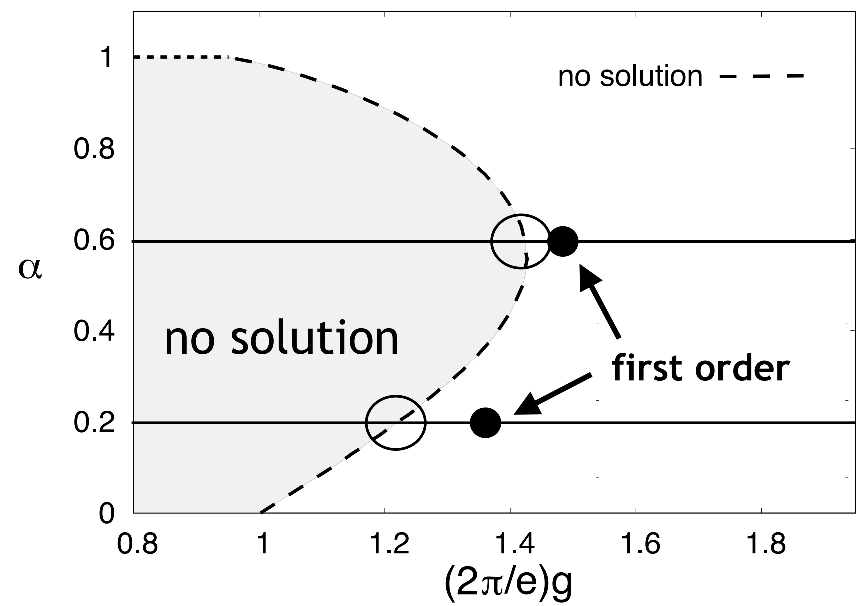

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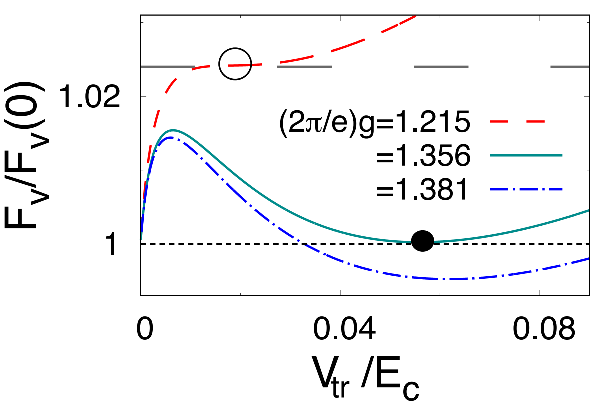
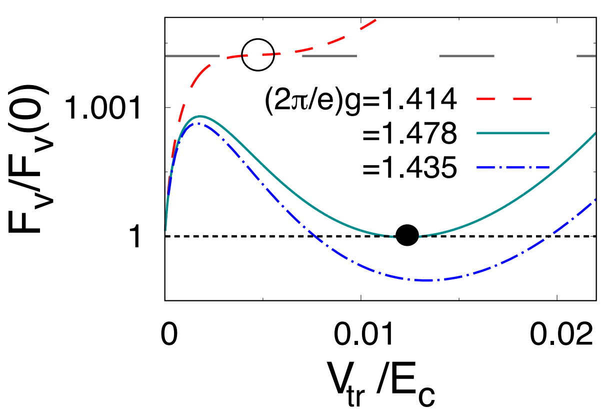


$\alpha = 0.6$



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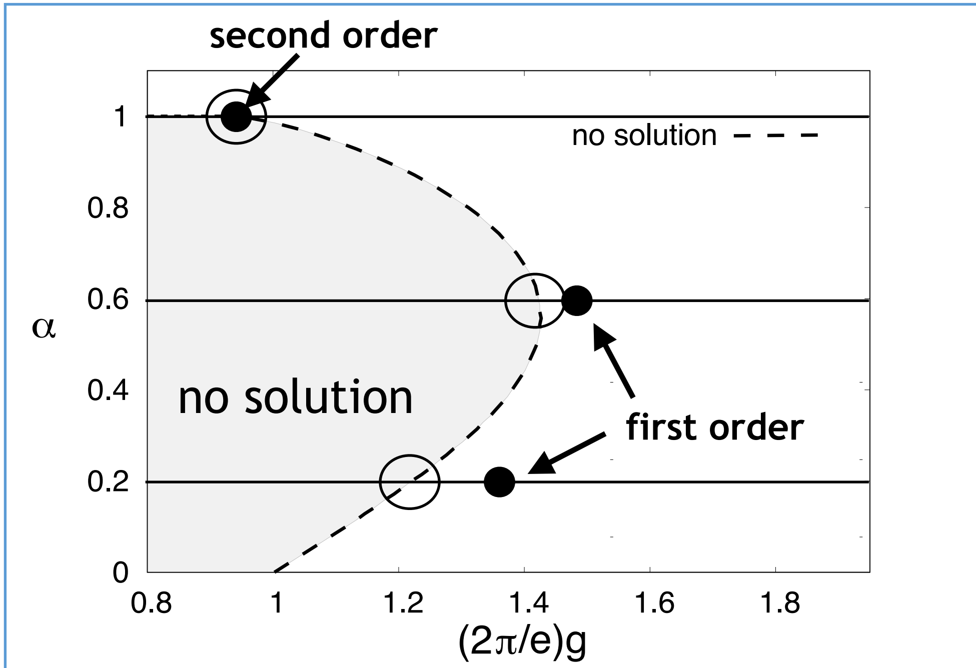

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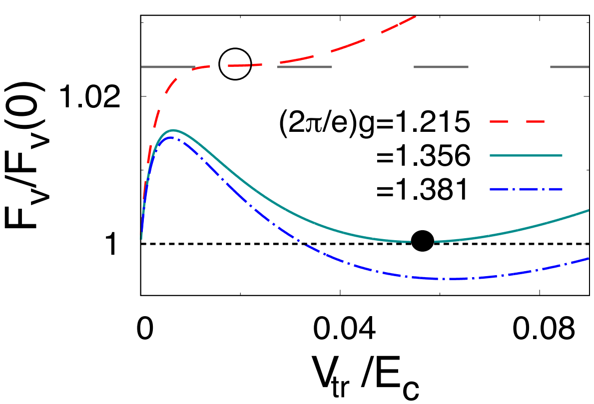

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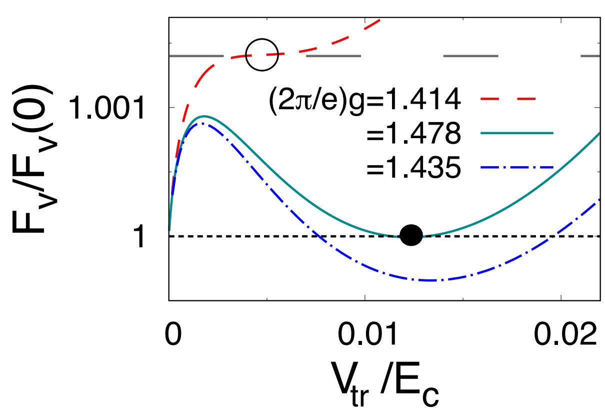
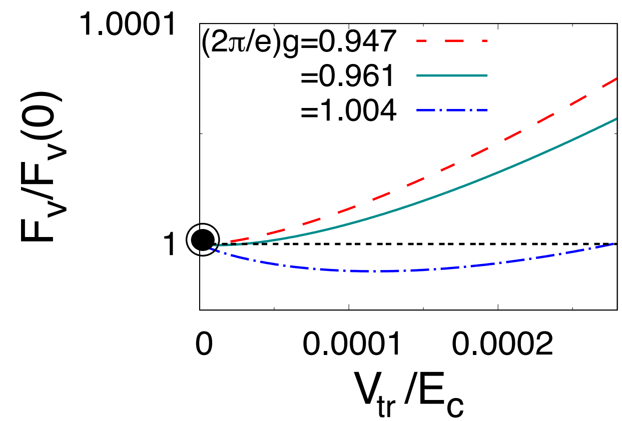


$\alpha = 1.0$



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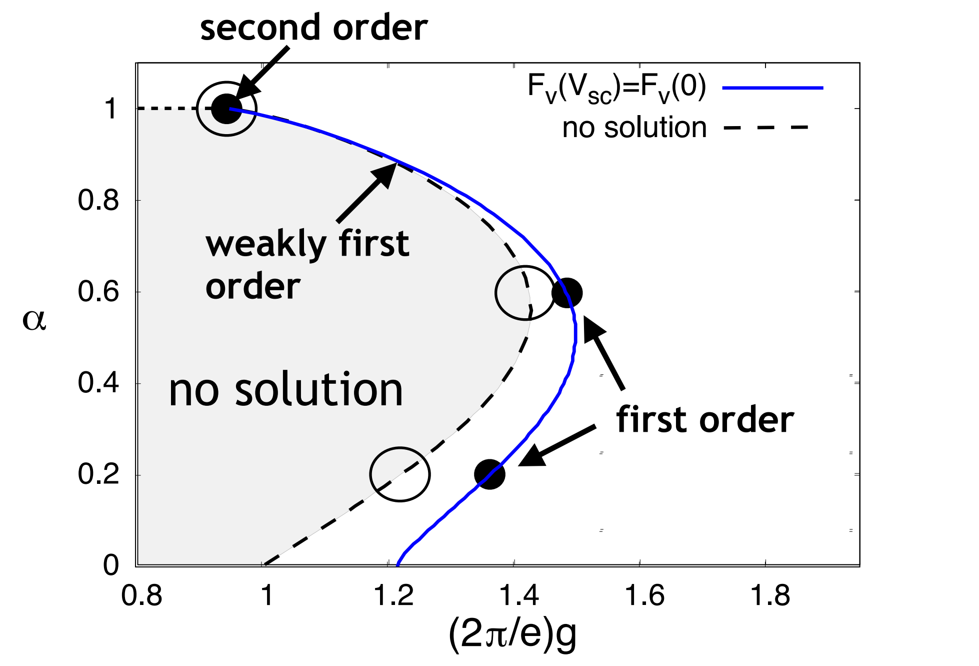

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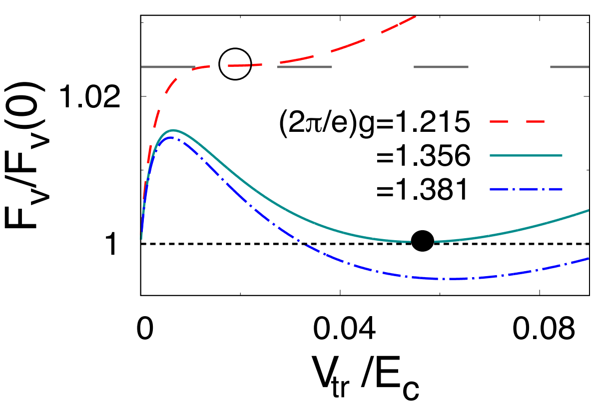
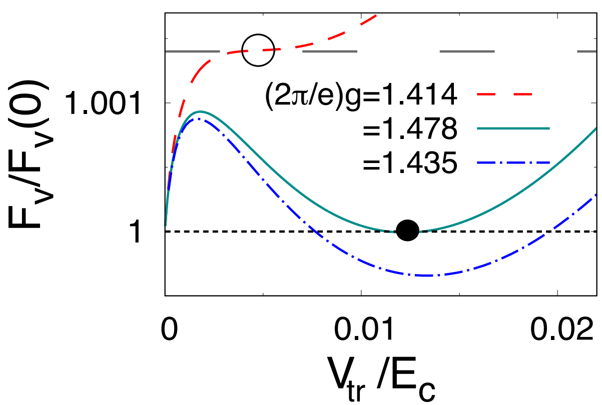
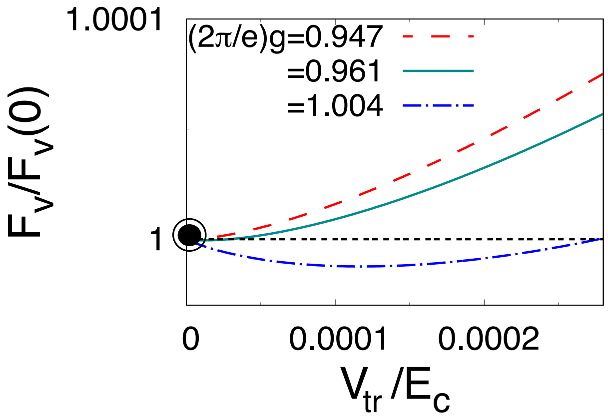


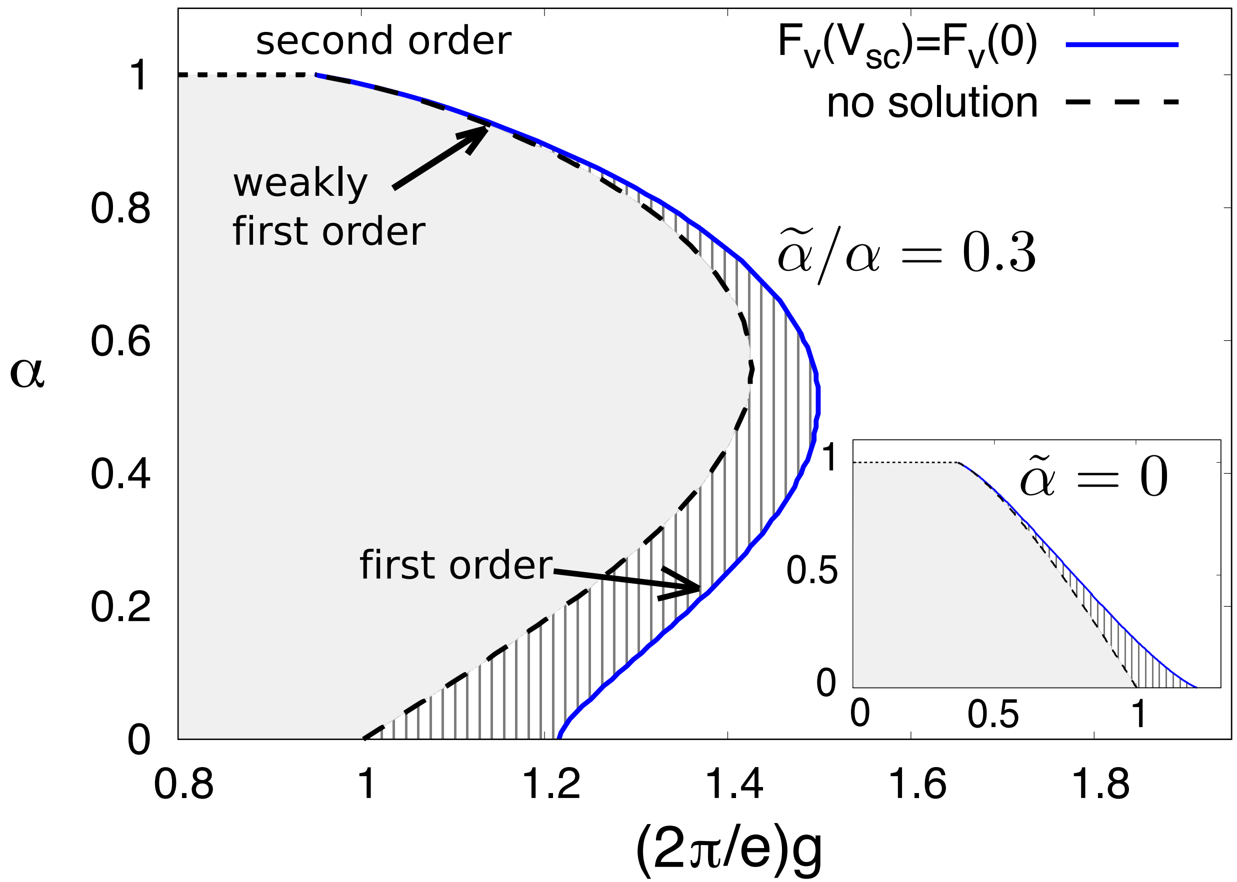
$\alpha = 0.6$



$\alpha = 1.0$



Phase diagram



→ Dissipative frustration leads to a non-monotonic phase diagram

Quantum thermodynamic quantities

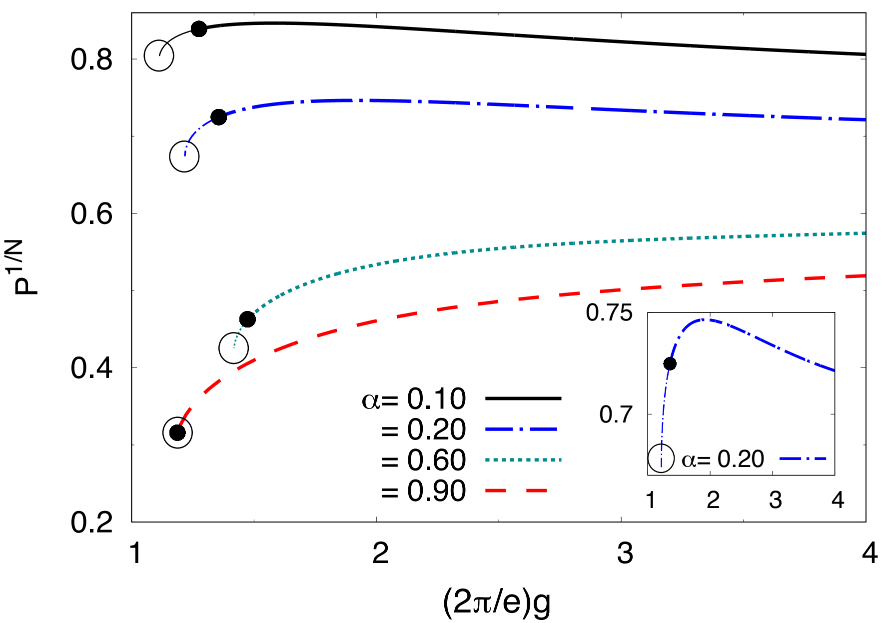
System-environment correlation

Purity

$$P = \mathrm{tr}\left(\hat{\rho}^2\right) = \prod_k \sqrt{\frac{\langle|\varphi_k|^2\rangle_0 \langle|\dot{\varphi}_k|^2\rangle_0}{\langle|\varphi_k|^2\rangle \langle|\dot{\varphi}_k|^2\rangle}}$$

$\langle|\varphi_k|^2\rangle$ phase fluctuations with dissipation

$\langle|\varphi_k|^2\rangle_0$ phase fluctuations $\alpha = \tilde{\alpha} = 0$

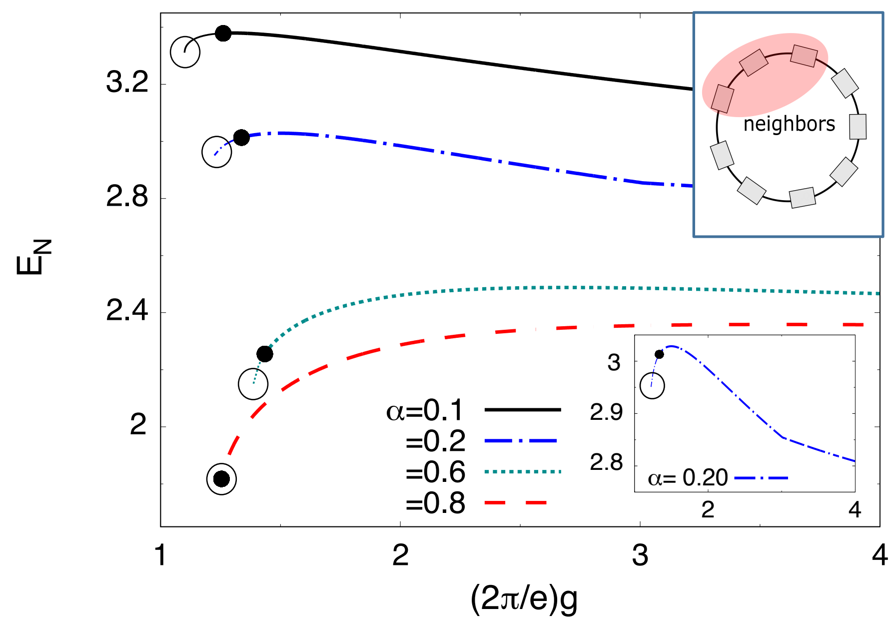


Quantum correlation | Entanglement

Logarithmic negativity

$$E_N(\hat{\rho}) = \log_2\left(1 - 2\sum_{\lambda_k<0} \lambda_k\left[\hat{\rho}^{T_A}\right]\right)$$

negative eigenvalues of the partially transposed density matrix

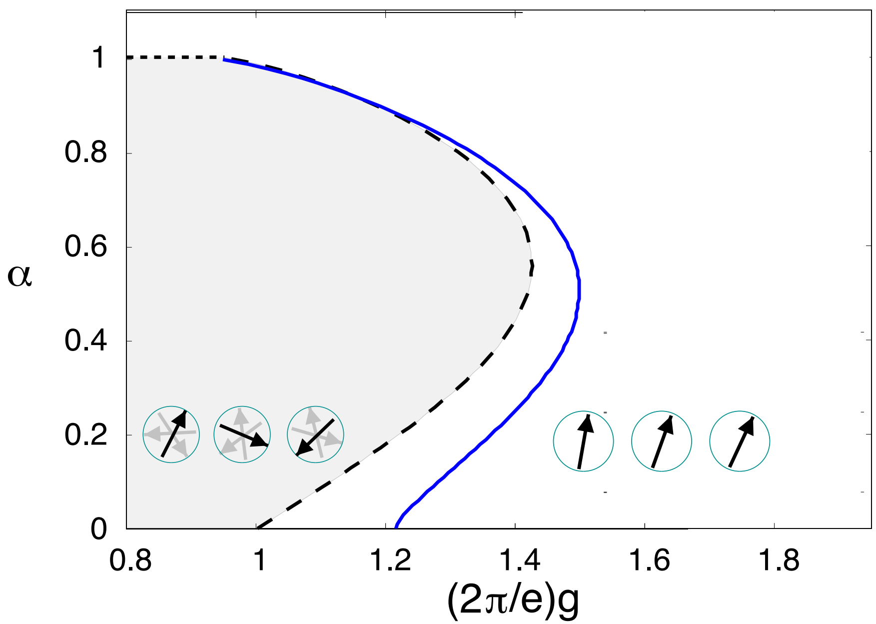


→ peaks close to the critical point only for $\alpha \neq 0$ and $\tilde{\alpha} \neq 0$

Summary

- In the quantum phase model realized by JJ chains with tailored dissipation, **dissipative frustration leads to a non-monotonic phase diagram**

- The purity and the logarithmic negativity show a peculiar behavior close to the critical point in presence of dissipative frustration



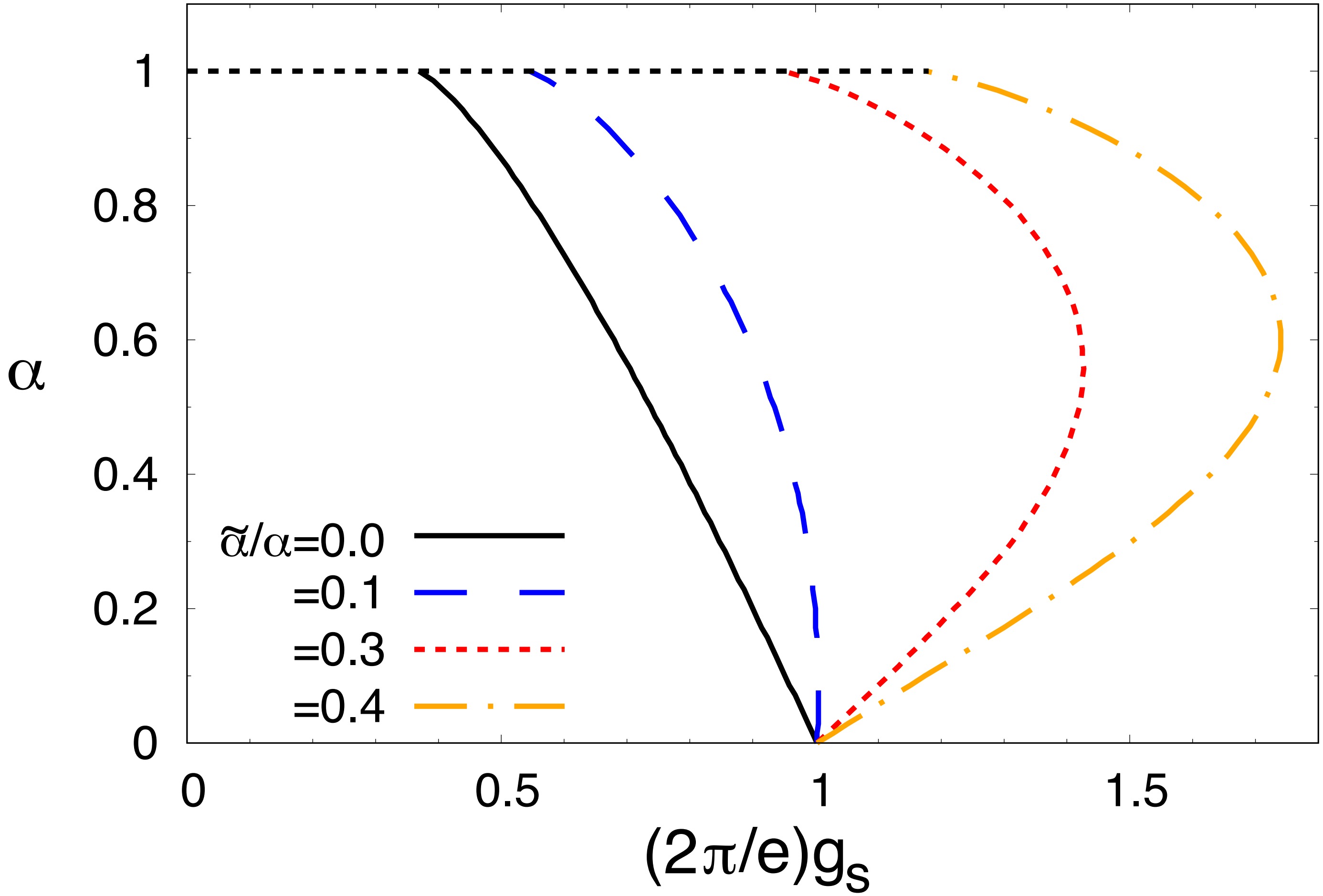
Quantum phase transition with dissipative frustration

arXiv:1711.11346

D. Maile, S. Andergassen, W. Belzig, G. Rastelli

Thank you

Different ratio of dissipative couplings $\widetilde{\alpha}/\alpha$



Fluctuations of the phase difference

$$\langle \Delta\varphi^2 \rangle_{sc} = \frac{4}{N} \sum_{k=1}^{N-1} \sin^2\left(\frac{\pi k}{N}\right) \langle \phi_k^2 \rangle_{sc}$$

$$\sigma_k^2 = 4\sin^2\left(\frac{\pi k}{N}\right)\gamma\tau_p,$$

$$\omega_{k,sc} = g\sqrt{V_{sc}V} \cdot 2 \cdot \sin\left(\frac{\pi k}{N}\right)$$

Harmonic Modes:

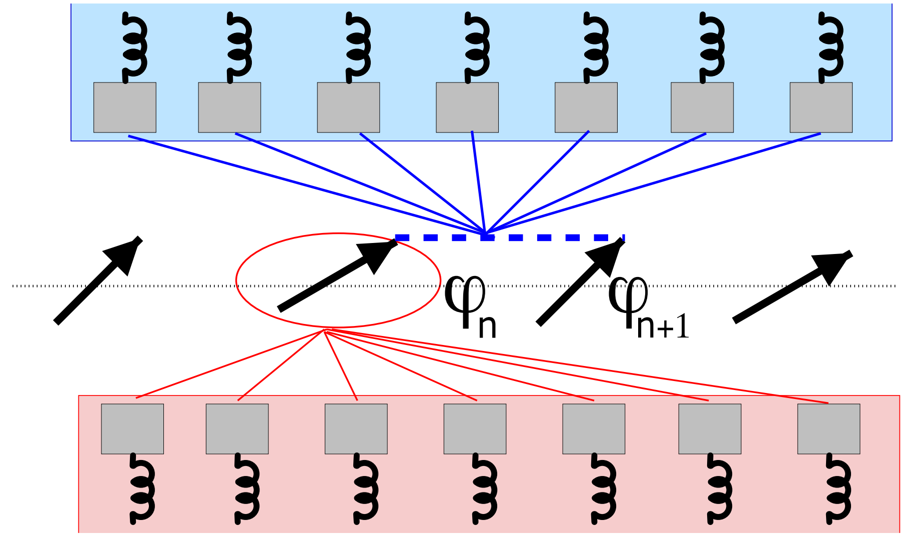
$$\langle \phi_k^2 \rangle_{sc} = \frac{g^2}{\hbar\pi}\frac{V}{1+\sigma_k^2}\left[\tau_p\left(\ln\left(\frac{\omega_c}{\omega_{k,sc}}\right) + \ln(1+\sigma_k^2) + \sigma_k \arctan(\sigma_k)\right) + G_k\right]$$

$$G_k = \left(\frac{1}{\omega_{k,sc}} + \tau_p \Delta\Gamma_{k,sc}\right)\begin{cases} \frac{1}{\sqrt{1-\Delta\Gamma_{k,sc}^2}}\arctan\left(\frac{\sqrt{1-\Delta\Gamma_{k,sc}^2}}{\Gamma_{k,sc}}\right) & \text{, for } \Delta\Gamma_{k,sc} < 1 \\ \frac{1}{\sqrt{\Delta\Gamma_{k,sc}^2-1}}\operatorname{arctanh}\left(\frac{\sqrt{\Delta\Gamma_{k,sc}^2-1}}{\Gamma_{k,sc}}\right) & \text{, for } \Delta\Gamma_{k,sc} > 1 \end{cases}$$

$$\Gamma_{k,sc} = \frac{4\sin^2\left(\frac{\pi k}{N}\right)\gamma + \omega_{k,sc}^2\tau_p}{2\omega_{k,sc}}$$

$$\Delta\Gamma_{k,sc} = \frac{4\sin^2\left(\frac{\pi k}{N}\right)\gamma - \omega_{k,sc}^2\tau_p}{2\omega_{k,sc}}$$

$$
\begin{aligned}
\widehat{H} &= \sum_{n=1}^{N-1}\left[\frac{\widehat{p}_n^2}{2m} - V\cos\left(\widehat{\varphi}_{n+1}-\widehat{\varphi}_n\right)\right] \\
&+ \sum_{n=1}^{N-1}\sum_{\lambda}\left[\frac{\widehat{P}_{n,\lambda}^2}{2M_\lambda} + \frac{M_\lambda\Omega_\lambda^2}{2}\left(\widehat{X}_{n,\lambda} - \frac{\alpha_\lambda R}{M_\lambda\Omega_\lambda^2}\left(\widehat{\varphi}_{n+1}-\widehat{\varphi}_n\right)\right)^2\right] \\
&+ \sum_{n=1}^{N-1}\sum_{m}\left[\frac{(\widehat{\widetilde{P}}_{n,m} - K_m\widehat{p}_n)^2}{2M_m} + \frac{M_m\Omega_m^2}{2}\widehat{\widetilde{X}}_{n,m}^{\,2}\right] \\
&= \widehat{H}_{sys} + \widehat{H}_{bath,1} + \widehat{H}_{bath,2}
\end{aligned}
$$



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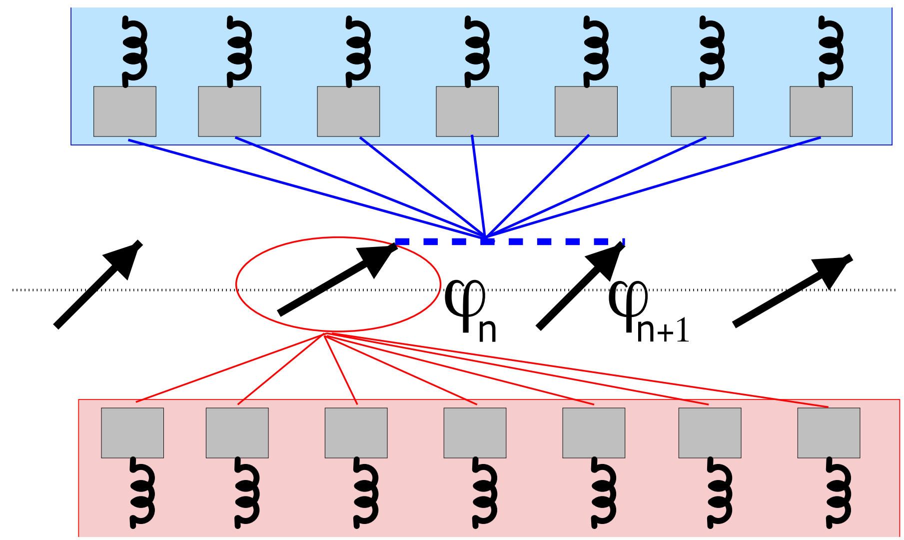

$$
\begin{aligned}
\widehat{H} \quad = \quad & \sum_{n=1}^{N-1}\left[\frac{\widehat{p}_n^2}{2m} - V\cos\left(\widehat{\varphi}_{n+1} - \widehat{\varphi}_n\right)\right] \\
& + \sum_{n=1}^{N-1}\sum_{\lambda}\left[\frac{\widehat{P}_{n,\lambda}^2}{2M_\lambda} + \frac{M_\lambda \Omega_\lambda^2}{2}\left(\widehat{X}_{n,\lambda} - \frac{\alpha_\lambda R}{M_\lambda \Omega_\lambda^2}\left(\widehat{\varphi}_{n+1} - \widehat{\varphi}_n\right)\right)^2\right] \\
& + \sum_{n=1}^{N-1}\sum_{m}\left[\frac{\left(\widehat{\widetilde{P}}_{n,m} - K_m \widehat{p}_n\right)^2}{2M_m} + \frac{M_m \Omega_m^2}{2}\widehat{\widetilde{X}}_{n,m}^{\,2}\right] \\
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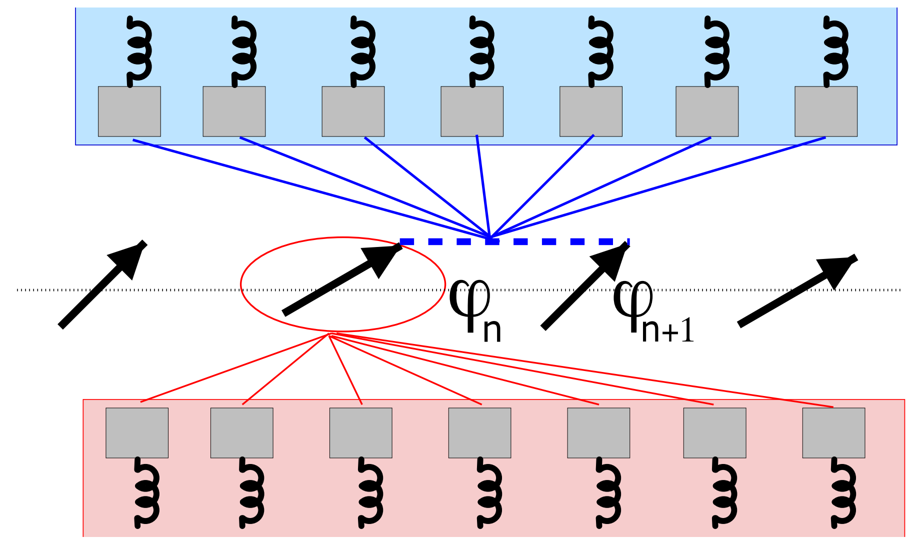
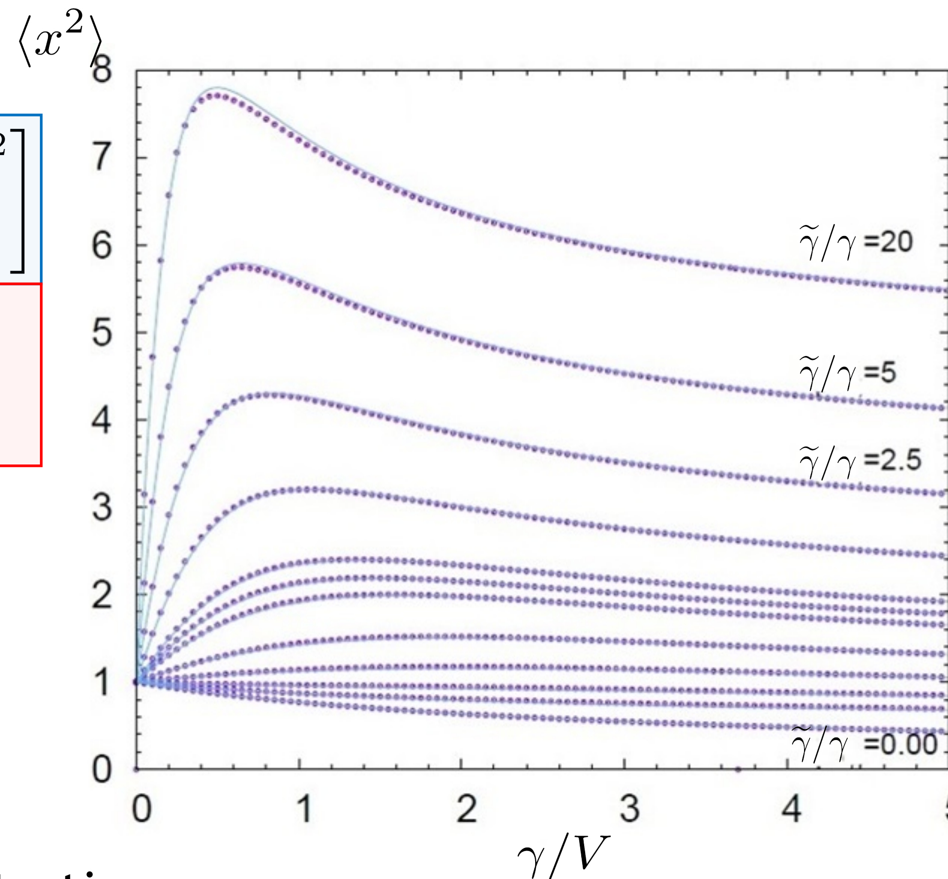


Quantum frustration

Harmonic oscillator coupled to two baths:

[G. Rastelli, New J. Phys. **18**, 053033 (2016)]

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{V}{2}\widehat{x}^2 + \sum_\lambda \left[\frac{\widehat{P}_\lambda^2}{2M_\lambda} + \frac{M_\lambda \Omega_\lambda^2}{2} \left(\widehat{X}_\lambda - \frac{\alpha_\lambda R}{M_\lambda \Omega_\lambda^2}\widehat{x} \right)^2 \right] + \sum_m \left[\frac{(\widehat{\widetilde{P}}_m - K_m \widehat{p})^2}{2M_m} + \frac{M_m \Omega_m^2}{2} \widehat{\widetilde{X}}_m^2 \right]$$



$$\delta x \cdot \delta p \geq \frac{\hbar}{2}$$ → quantum frustration

The effective action

$$S_{eff} = -\int_0^\beta d\tau \sum_{n=0}^{N-1} V \cos\left(\varphi_{n+1}(\tau) - \varphi_n(\tau)\right)$$

$$+ \frac{R^2}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \sum_{n=0}^{N-1} F(\tau_1 - \tau_2) \Delta\varphi_n(\tau_1) \Delta\varphi_n(\tau_2)$$

$$+ \frac{R^2}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \sum_{n=0}^{N-1} \widetilde{F}(\tau_1 - \tau_2) \dot{\varphi}_n(\tau_1) \dot{\varphi}_n(\tau_2)$$

$$\Delta\varphi_n = (\varphi_{n+1} - \varphi_n)$$

Matsubara transformation:

$$F(\tau) = \sum_{l=-\infty}^{\infty} F_l e^{i\omega_l \tau}, \quad F_l = \frac{\gamma m |\omega_l|}{\beta \left(1 + \frac{|\omega_l|}{\omega_c}\right)}$$

$$\omega_l = \frac{2\pi}{\beta} l \qquad \beta = \frac{\hbar}{k_B T}$$

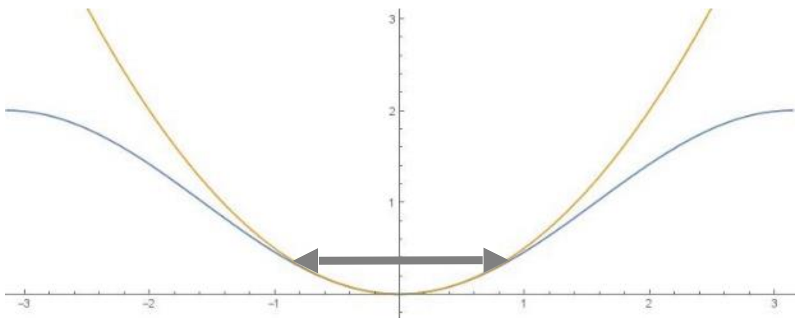
$$\widetilde{F}(\tau) = \sum_{l=-\infty}^{\infty} \widetilde{F}_l e^{i\omega_l \tau}, \quad \widetilde{F}_l = \frac{m}{\beta \left[1 + \tau_p \frac{|\omega_l|}{\left(1 + \frac{|\omega_l|}{\omega_c}\right)}\right]}$$

$$\widetilde{\gamma} = \tau_p V^2$$

Self-Consistent Harmonic Approximation

Bogoliubov inequality for free energy:

$$F \leq \underbrace{-k_B T \ln(Z_s) - \frac{1}{\beta}\langle \Delta S \rangle_s}_{F^{(1)}}$$



$$\Delta S = S_{eff} - S_s$$

$$= -\int_0^\beta d\tau \sum_{n=0}^{N-1} V \cos\left(\varphi_{n+1}(\tau) - \varphi_n(\tau)\right) - \int_0^\beta d\tau \sum_{n=0}^{N-1} \frac{V_s}{2} \left(\varphi_{n+1}(\tau) - \varphi_n(\tau)\right)^2$$

$$\frac{dF^{(1)}}{dV_s} \stackrel{!}{=} 0 \quad \longrightarrow \quad V_s = V e^{-\frac{1}{2}\langle \Delta\varphi^2 \rangle_s}$$

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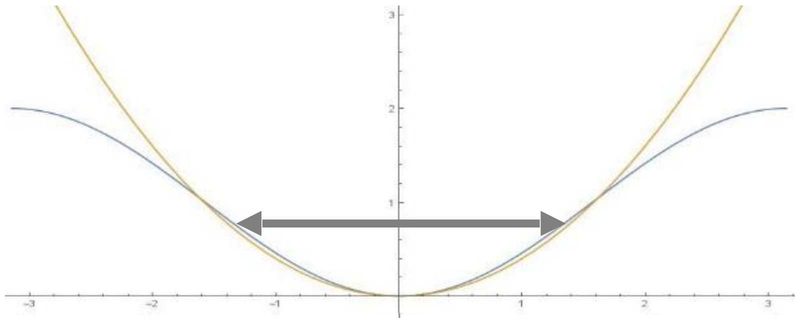

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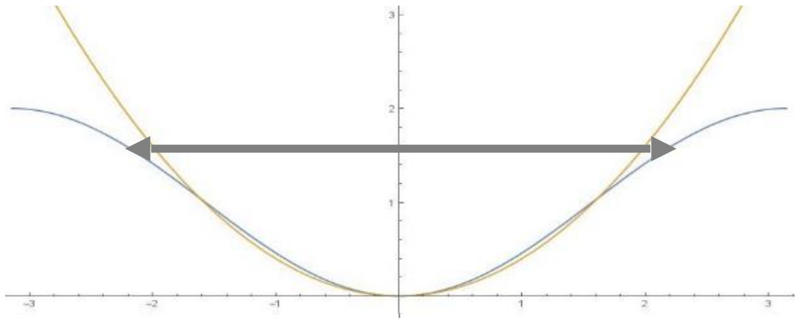

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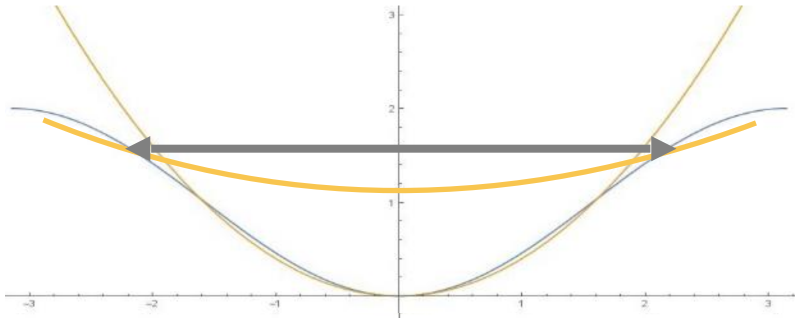

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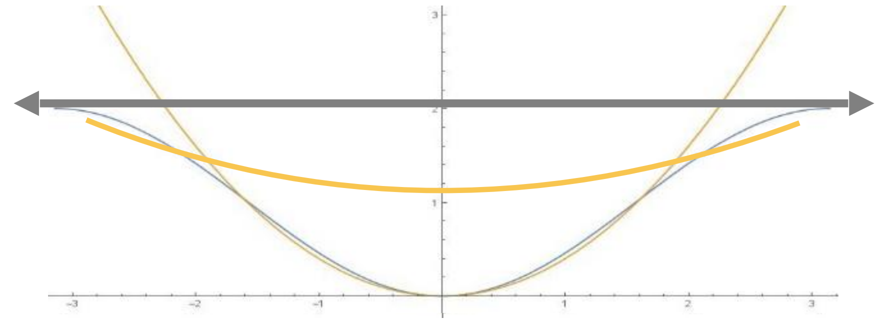

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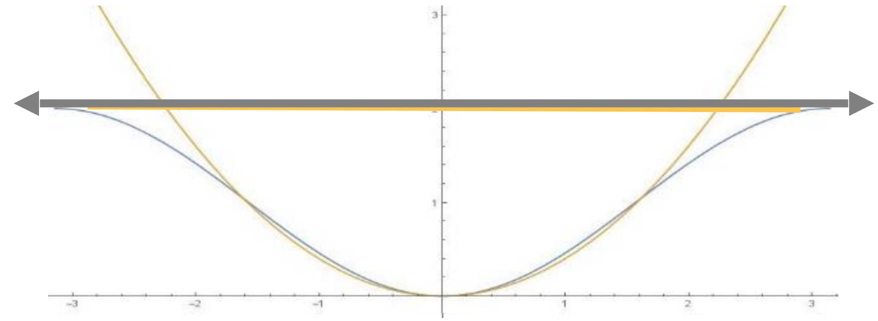

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$$\widetilde{F}(\tau) = \sum_{l=-\infty}^{\infty} \widetilde{F}_l e^{i\omega_l \tau}, \; \widetilde{F}_l = \frac{m}{\beta \left[1 + \tau_p \frac{|\omega_l|}{\left(1 + \frac{|\omega_l|}{\omega_c}\right)}\right]}$$

$$\alpha = R_q/R_s = \gamma(h/E_C) \quad \text{with} \;\; \gamma = 1/(R_s C_0)$$

$$\widetilde{\alpha} = R_g/R_q = \tau_g(E_C/h) \quad \text{with} \;\; \tau_p = R_g C_0$$

The partition function

Definition of the free energy:

$$F = -k_B T \ln(Z)$$

Definition of the partition function:

$$Z = \mathrm{tr}\left(e^{-\frac{\beta}{\hbar}\hat{H}}\right)$$

Imaginary path integral representation:

$$Z = \prod_{n=0}^{N-1} \int RD[\varphi_n(\tau)] \prod_{\lambda} \int D[X_{n,\lambda}(\tau)] \prod_{m} \int D[\widetilde{X}_{n,m}(\tau)]\, e^{-\frac{1}{\hbar}S}$$

$$Z = Z_{bath,1}^{N} Z_{bath,2}^{N} \prod_{n=0}^{N-1} \oint_{\substack{\varphi(0)=\varphi_0 \\ \varphi(\beta)=\varphi_0+2\pi k_n}} RD[\varphi_n(\tau)]\, e^{-\frac{1}{\hbar}S_{eff}}$$

$$\varphi \in [-\pi, \pi] \to (-\infty, \infty)$$