Majorana fermions and their generalizations in CM

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I. PLAN

In these lectures, I will touch upon these topics:

1. A Few words about topological order.
2. p-wave superconductors
3. Kitaev chain
4. Experimental platforms: TI surfaces, nanowires, 2d systems.
5. Signatures: conductance, spin, topological Josephson Junctions.
6. Topological superconductivity in a planar Josephson junction (Yacoby's experiment)
7. Beyond Majorana

II. TOPOLOGICAL STATES OF MATTER

Most broad definition: gapped quantum states of matter that do not spontaneously break any symmetry of the Hamiltonian, yet they are distinct from a trivial "atomic" insulator. There is necessarily a phase transition between the two phases. "Hidden" (non-local or "topological") order encoded in the ground state wavefunction. Often (but not always), these states have topologically protected gapless edge states, that cannot be removed unless the bulk is destroyed (i.e., undergoes a phase transition to a different phase).

Examples:

1. Topological insulators in 2D and 3D, Haldane $S = 1$ chain. These are "symmetry protected topological phases": they are distinct from an atomic band insulator as long as some symmetry is maintained.
2. The integer quantum Hall effect.

A further useful distinction to make is between *topologically ordered* states and other "topological" states. A prime example of a topologically ordered state is the fractional quantum Hall state in $d = 2$. The topological ordered ones are characterized by several common features:

1. These states support collective "point-like" excitations (quasi-particles) that obey fractional statistics, i.e., they are *anyons.* These excitations often (but not always) carry fractional quantum numbers.

2. The ground state has a non-trivial, topologically protected degeneracy that depends on the topology of the manifold on which the system lives. The ground state wavefunction is able to “sense” the topology of the surface.

Perhaps the most interesting of these states are *non-Abelian* states, in which there is a topologically protected degeneracy that grows exponentially with the number of quasi-particle excitations. Each such quasi-particle has its own characteristic degeneracy. Exchange of two quasi-particles implements a unitary matrix on the degenerate subspace, and different unitary matrices do not commute, hence the name. Topologically ordered states can only appear in strongly interacting systems.

In these lectures, we will discuss topological states of matter that do not exactly fit into either of these categories. The states we will discuss appear in superconductors, and can (for the most part) be described in terms of non-interacting, quadratic Hamiltonians (more precisely, to describe them, it is enough to consider the interactions at the mean field level). Nevertheless, they have some of the characteristics that we associate with topological order: they are *not* symmetry protected, and they can have topologically protected ground state degeneracies of non-local origin. These degeneracies are not related to the topology of the manifold the system lives on. Rather, they are associated with *defects*, such as edges, vortices, or dislocations. Moreover, there is a well-defined sense in which “braiding” can be performed (even though we are not dealing with point particles), and the result of the braiding operation is a topologically protected non-Abelian operation.

Over the last decade, there has been significant progress in realizing some of these systems experimentally. We will discuss the experimental platforms and the physical signatures of these states.

III. BDG FORMALISM AND P-WAVE SUPERCONDUCTORS.

$$H = \sum_{i,j} - \overbrace{t_{ij}\ \psi_i^{+}}^{\text{hopping}} \psi_j - \overbrace{\mu \sum_i \psi_i^{+}}^{\text{Chemical potential}} \psi_i + \overbrace{\sum_{i,j} \Delta_{ij}}^{\text{pairing potential}} c_i^{+} c_j^{+} + h.c.$$

$$= \underbrace{(\psi_1^\dagger \dots \psi_N^\dagger \quad \psi_1 \dots \psi_N)}_{\Psi^\dagger} \underbrace{\begin{pmatrix} h_{ij} & \Delta_{ij} \\ (\Delta^\dagger)_{ij} & -h_{ij}^* \end{pmatrix}}_{\mathcal{H} : 2N \times 2N} \left.\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^\dagger \\ \vdots \\ \psi_N^\dagger \end{pmatrix}\right\} \Psi$$

Nambu Spinor (Here $h_{ij} = \frac{1}{2}(-t_{ij} - \mu \delta_{ij})$)

Quadratic in ψ's → can be solved!

Bugoliubov transformation:

$$\begin{pmatrix} \psi_1 \\ | \\ \psi_N \\ \psi_1^\dagger \\ | \\ \psi_N^\dagger \end{pmatrix} = \overset{\text{Unitary}}{U} \left.\begin{pmatrix} f_1 \\ | \\ f_N \\ f_1^\dagger \\ | \\ f_N^\dagger \end{pmatrix}\right\} F$$

$$\boxed{\text{Check:} \quad \{f_i^\dagger, f_j\} = \delta_{ij}}$$

$$H = \Psi^\dagger \mathcal{H} \Psi = F^\dagger(\underbrace{U^\dagger \mathcal{H} U}) F$$

Choose U to diagonalize $\mathcal{H}$:

$$U^\dagger \mathcal{H} U = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_{2N} \end{pmatrix}$$

Symmetry of the BdG Hamiltonian:

$$C^{-1} \mathcal{H} C = -\mathcal{H}, \quad \text{where} \quad C = \begin{pmatrix} 0 & \mathbb{1}_{N\times N} \\ \mathbb{1}_{N\times N} & 0 \end{pmatrix} \underbrace{K}_{\text{Complex conjugation}}$$

"Particle - hole" symmetry

(You can prove this by direct substitution

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K = K \begin{pmatrix} -h^* & \Delta^\dagger \\ \Delta & h \end{pmatrix} K$$

$$= \begin{pmatrix} -h & \Delta^T \\ \Delta^* & h^* \end{pmatrix}$$

You can check that $\Delta^T = -\Delta$

$$\Delta^* = (\Delta^T)^\dagger = -\Delta^\dagger$$

since if Δ is symmetric, $\Delta_{ij} c_i^\dagger c_j^\dagger = 0$

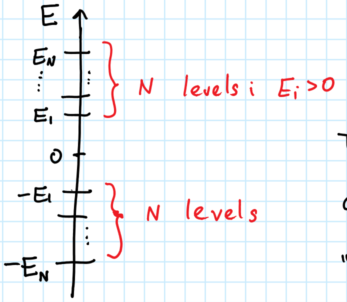
$$\Rightarrow \dots = -\begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix}.$$)

Note also that $\Psi = \begin{pmatrix} \psi_1 \\ | \\ \psi_N \\ \psi_1^\dagger \\ | \\ \psi_N^\dagger \end{pmatrix} = C \begin{pmatrix} \psi_1^\dagger \\ | \\ \psi_N^\dagger \\ \psi_1 \\ | \\ \psi_N \end{pmatrix} \equiv C\, \Psi^*$.

This implies that the levels of $\mathcal{H}$ come in pairs of opposite sign: if $\mathcal{H} u = E u$, then

$$\mathcal{H}(C u) = C(\underbrace{C^{-1} \mathcal{H} C}_{-\mathcal{H}})\, u$$

$$= -E\, C u$$



The levels at E_i and $-E_i$ are not independent: they are "two halves of the same state".

After the transformation: $\mathcal{H} = F^\dagger \begin{pmatrix} E_1 & & & & & \\ & \ddots & & & & \\ & & E_N & & & \\ & & & -E_1 & & \\ & & & & \ddots & \\ & & & & & -E_N \end{pmatrix} F$

$$= \sum_{i=1}^{N} E_i \left(f_i^\dagger f_i - f_i f_i^\dagger \right)$$

$$= \sum_{i=1}^{N} E_i \left(2 f_i^\dagger f_i - 1 \right)$$

Many - body ground state: $f_i^\dagger f_i |0\rangle = 0 \quad \forall \quad i = 1 \ldots N$

(Since $E_i > 0$)

Excited states: $\prod_{j=1}^{n} f^\dagger_{\alpha_1} \ldots f^\dagger_{\alpha_n} |0\rangle$

Energy (rel. to ground state

$2(E_{\alpha_1} + .. + E_{\alpha_n})$

1d p-wave Hamiltonian:

$$H = \left(-2t\cos k - \mu\right) c_k^\dagger c_k + \Delta \sin k_x c_k^\dagger c_{-k}^\dagger + h.c. \tag{1}$$

Phase transition at $\mu = \pm 2t$, at $k_x = 0$, $k_x = \pi$, respectively.

Change μ slowly in space. Write $\delta\mu = \mu - 2t$. For small k, expand to first order in kxs around $k = 0$:

$$H \approx \begin{pmatrix} -\delta\mu(x) & -i\Delta\partial_x \\ -i\Delta\partial_x & \delta\mu(x) \end{pmatrix} \tag{2}$$

Write as

$$H = -\delta\mu(x)\tau^z - \Delta\tau^x i\partial_x. \tag{3}$$

Look for zero energy solution:

$$\left[-\delta\mu(x)\tau^z - i\Delta\tau^x\partial_x\right]\psi = 0 \tag{4}$$

Multiply by $i\tau^x$:

$$\left[\delta\mu(x)\tau^y + \Delta\partial_x\right]\psi = 0 \tag{5}$$

$$\psi = e^{-\int_0^x dx' \frac{\delta\mu(x)\tau^y}{\Delta}} \tag{6}$$

This can be normalized at $\pm\infty$ (with an appropriate choice of the eigenvalue of τ^y) if $\mu(\pm\infty)$ have opposite signs.

(From here, we can derive the dispersing edge states of the 2d case, and the vortex states.)

Derive the fact that the ground state parity is opposite for the two b.c.'s in one phase, and the same in another phase, by looking at quantized momenta. Relate this to the existence of the zero modes.

Discuss the ground state degeneracy, and how to construct it.

Kitaev's chain:

$$H = \sum_j -tc_j^\dagger c_{j+1} + \Delta c_j^\dagger c_{j+1}^\dagger + H.c. - \mu c_j^\dagger c_j \tag{7}$$

Write in terms of Majorana operators:

$$c_j = \frac{\alpha_j + i\beta_j}{2} \tag{8}$$

$$\begin{aligned} \alpha_j &= c_j + c_j^\dagger \\ \beta_j &= c_j - ic_j^\dagger \end{aligned} \tag{9}$$

$$\alpha^\dagger = \alpha,\ \beta^\dagger = \beta,\ \{\alpha_i, \beta_j\} = \delta_{ij}$$

$$\begin{aligned}H &= \sum_j -t(\alpha_j - i\beta_j)(\alpha_{j+1} + i\beta_{j+1}) + \Delta(\alpha_j - i\beta_j)(\alpha_{j+1} - i\beta_{j+1}) + H.c. - \mu(\alpha_j - i\beta_j)(\alpha_j + i\beta_j) \\ &= \sum_j -2t(i\alpha_j\beta_{j+1} - i\beta_j\alpha_{j+1}) + 2\Delta(-i\alpha_j\beta_{j+1} - i\beta_j\alpha_{j+1}) - \mu(i\alpha_j\beta_j - i\beta_j\alpha_j) \\ &= \sum_j -(2t + 2\Delta)i\alpha_j\beta_{j+1} + (2t - 2\Delta)i\beta_j\alpha_{j+1} - \mu(i\alpha_j\beta_j - i\beta_j\alpha_j). \end{aligned} \tag{10}$$

Majoranas at the ends:

$$\gamma^\dagger_{R,L} = \gamma_{R,L}. \tag{11}$$

Argue for: 1. The phase is characterized by the change in the parity of the ground state when a flux is inserted. 2. Phase transition (in non-interacting system) must occur by closing and re-opening of the gap at either $k = 0$ or at $k = \pi$. (g.s. parity has to switch at one of these points.)

If we have two phases, one where the parity of the ground state changes when we insert a flux and one where it does not, then we cannot go adiabatically between the two. This is true even in the presence of interactions. This is because, as we go from one phase to the other, then at least in one boundary condition sector, the ground state parity would have to switch, which implies that the gap has to close.

Key properties:

1. Ground state is two fold degenerate. $[\gamma_{L,R}, H] = 0$, $[i\gamma_L\gamma_R, H] = 0$.
2. $i\gamma_L\gamma_R$ is the total fermion parity of the ground state.
3. No local way to detect which ground state the system is in. (*Any* local observable is the same in the two ground states!)

IV. EXPERIMENTAL REALIZATIONS AND SIGNATURES

Rule of thumb: whenever we have a single Fermi surface in the normal state, if we manage to gap it, we will get a topological superconducting state.

Example: Fu-Kane superconductor, in either 1d or 2d.

Oreg-Lutchyn-von Oppen-Refael-Sau-das Sarma wire:

Necessary ingredients: 1. Spin-orbit coupling. 2. Breaking of time reversal symmetry. 3. superconductivity.

$$H = c_k^\dagger \left(\frac{k_x^2}{2m} - \alpha k_x \sigma^y - \mu - B\sigma^x\right) c_k + \Delta c^\dagger_{k\uparrow} c^\dagger_{k\downarrow} \tag{12}$$

Write BdG Hamiltonian:

$$H = \begin{pmatrix} c^\dagger_{k\uparrow} & c^\dagger_{k\downarrow} & c_{-k\downarrow} & -c_{-k\uparrow} \end{pmatrix} \begin{pmatrix} \frac{k_x^2}{2m} - \alpha k_x \sigma^y - \mu - B\sigma^x & \Delta \\ \Delta & -(\frac{k_x^2}{2m} - \alpha k_x\sigma^y - \mu) - B\sigma^x \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\downarrow} \\ -c^\dagger_{-k\uparrow} \end{pmatrix}$$

Hamiltonian at $k=0$:

$$H=\mu\tau^z+\Delta\tau^x-B\sigma^x.$$

$$\pm\left(\sqrt{\Delta^2+\mu^2}\pm B\right)$$

Explain this pictorially. Similarly for 2d: obtain analogue of the p+ip superconductor.

Mention also magnetic adatoms on Pb surface and STM.

Transport signature: zero bias peak. Particle-hole symmetry dictates that we either have perfect normal reflection, or perfect Andreev reflection at $E=0$. Derive:

$$\psi_{out}=R\psi_{in}$$

$$\psi=\begin{pmatrix}u(x)\\v(x)\end{pmatrix},$$

write R as:

$$R=\begin{pmatrix}r_{ee}&r_{he}\\r_{eh}&r_{hh}\end{pmatrix}$$

solution to the SE:

$$\psi_e(x)=\begin{pmatrix}1\\0\end{pmatrix}e^{ikx}+R\begin{pmatrix}1\\0\end{pmatrix}e^{ikx}=\begin{pmatrix}1\\0\end{pmatrix}e^{ikx}+\begin{pmatrix}r_{ee}e^{-ikx}\\r_{eh}e^{ikx}\end{pmatrix}. \tag{13}$$

$$\psi_h(x)=\begin{pmatrix}0\\1\end{pmatrix}e^{-ikx}+R\begin{pmatrix}0\\1\end{pmatrix}e^{-ikx}=\begin{pmatrix}0\\1\end{pmatrix}e^{-ikx}+\begin{pmatrix}r_{he}e^{-ikx}\\r_{hh}e^{ikx}\end{pmatrix}. \tag{14}$$

Clearly, these two solutions have to be p-h conjugates. Therefore

$$r_{ee}=r^*_{hh}$$

$$r_{he}=r^*_{eh}$$

$$R=\begin{pmatrix}r_{ee}&r^*_{eh}\\r_{eh}&r^*_{ee}\end{pmatrix}. \tag{15}$$

Unitarity requires that

$$r_{ee}r_{eh}+r_{eh}r_{ee}=0$$

$$2r_{ee}r_{eh} = 0.$$

I.e., either $r_{ee} = 0$ or $r_{eh} = 0$. This implies that there is either only normal reflection, or only Andreev reflection.

Note also that

$$det(R) = |r_{ee}|^2 - |r_{eh}|^2\,, \tag{16}$$

I.e., the determinant is either -1 (in the topological phase) or 1 (trivial phase). This is general, e.g., it is true for any number of channels in the lead. However, in general, if there are multiple channels, the reflection does not have to be purely Andreev or purely normal. If there is only one "dominant" channel, then it might. Perfect Andreev reflection implies $G = 2e^2/h$. The Kouwenhoven group claims to have seen this experimentally.

4π periodic Josephson effect: if we have a weak link between two topological superconductors, we can describe it in terms of the coupling (or hopping) between the two Majorana end modes, $\gamma_{1,2}$. If the phase difference between the two superconductors is ϕ, The Josephson Hamiltonian has the form

$$H_J = i\Gamma e^{i\phi/2}\gamma_1\gamma_2 + H.c. = i\Gamma\gamma_1\gamma_2\cos(\phi/2). \tag{17}$$

Therefore, in principle, if we apply an ac voltage, that gives $\phi = \frac{2eV}{\hbar}t$, we get that the peridicity of the current (ac Josephson effect) is $\omega = \frac{eV}{\hbar}$, i.e., half of the usual periodicity. This has the caveat that, if we go too slowly, the usual periodicity would be recovered, because of "quasi-particle poisoning."

Another interesting consequence of the topological degeneracy between the two states is the fact, in a finite topological wire, there is no gap between even and odd parity states. This has been tested by the Marcus group, using a "Coulomb blockade" setup, where current is measured through the system. The energy of the system is of the form

$$E(N, V_B) = \frac{e^2(N-N_G)^2}{2C} + f(N). \tag{18}$$

where $N_G = CV_G$, the gate voltage. Usually in a superconductor,

$$f(N) = \frac{1-(-1)^N}{2}\Delta. \tag{19}$$

This leads to an even-odd effect in the spacing between the Coulomb blockade peaks. However, in the Majorana wire there is no even-odd effect in the limit of a long wire. This has been observed by the Marcus group. Even the decay of the even-odd splitting has been seen.

Another interesting consequence of having a Majorana zero mode is related to the spin. Naively, the Majorana operator at the edge can be written as

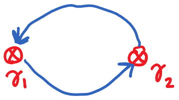
$$\gamma = ac_\uparrow + bc_\downarrow + a^*c_\uparrow^\dagger + b^*c_\downarrow^\dagger. \tag{20}$$

The spinor (a, b) corresponds to a given spin direction. hence, the Majorana wavefunction has a preferred spin. We might expect that the conductance at zero bias for the other spin flavor would be strongly suppressed; this has been claimed to have been seen in surfaces of TIs, proximitized with a superconductor. This argument is too naive in general, though, because it assumes that the contact between the superconductor and the lead is just at one spatial point; at each point, the direction of the polarization of the Majorana wavefunction can be different.

Another effect is related to the noise. If we have a Majorana zero mode, then each Andreev reflection is either with spin up or spin down, with probabilities $|a|^2$ and $|b|^2$, respectively, and it is completely independent of the previous one. This has to be the case, because if there was "memory", it would mean that we can "measure" the state of the wire - whether it is parity even or parity odd - by measuring the spin of the Andreev-reflected electron. This has interesting consequences for the current noise. Suppose we have a measurement over a time t, and we look at the fluctuations of the charge $\delta Q_\uparrow$, $\delta Q_\downarrow$. Since at low V the Andreev reflection becomes almost perfect, $\langle(\delta Q_\uparrow + \delta Q_\downarrow)^2\rangle \to 0$; therefore, $\langle \delta Q_\uparrow \delta Q_\downarrow \rangle < 0$. In contrast, for a regular Andreev bound state (equivalent to two Majorana zero modes), there is generically positive correlation between the spin up and spin down currents.

V. NON-ABELIAN PROPERTIES

Given $2N$ Majoranas, what is the effect of exchanging i and j?



How do the operators $\gamma_{1,2}$ transform under braiding?

$\gamma_1 \rightarrow \gamma_1' = U_{12}^\dagger \gamma_1 U_{12}$ U_{12} : Unitary adiabatic evolution operator.

$\gamma_2 \rightarrow \gamma_2' = U_{12}^\dagger \gamma_2 U_{12}$

We expect

$$\gamma_1' \propto \gamma_2 \quad \text{(up to a phase)}$$

$$\gamma_2' \propto \gamma_1$$

$(\gamma_1')^2 = U_{12}^\dagger \gamma_1 U_{12} U_{12}^\dagger \gamma_1 U_{12} = 1 \Rightarrow$ phases are ±1

suppose $\gamma_1' = \gamma_2$

transformation has to conserve $i\gamma_1\gamma_2$ (fermion parity of 1, 2)

$\rightarrow \quad \gamma_2' = -\gamma_1$!

One can check that the transformation that does this is

$$\boxed{U_{12} = \underbrace{e^{i\phi}}_{\text{phase we can't determine from the present considerations.}} e^{\frac{\pi}{4}\gamma_1\gamma_2}}$$

check: $e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2)$

$$e^{-\frac{\pi}{4}\gamma_1\gamma_2}\;\gamma_1\;e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{2}(1-\gamma_1\gamma_2)\,\gamma_1\,(1+\gamma_1\gamma_2)$$

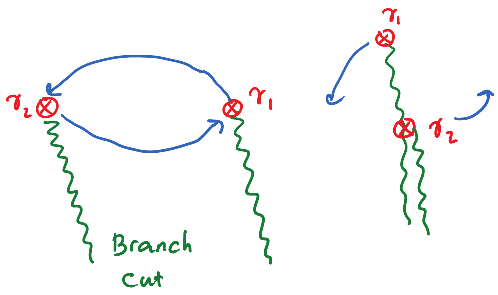
$$= \frac{1}{2}(\gamma_1+\gamma_2+\gamma_2-\gamma_1)$$

$$= \gamma_2$$

$$e^{-\frac{\pi}{4}\gamma_1\gamma_2}\;\gamma_2\;e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{2}(1-\gamma_1\gamma_2)\,\gamma_2\,(1+\gamma_1\gamma_2)$$

$$= \frac{1}{2}(\gamma_2-\gamma_1-\gamma_1-\gamma_2) = -\gamma_2$$

Ivanov (2001) argument: When a fermion goes around a vortex in a S.C., it gets a (−1) sign.



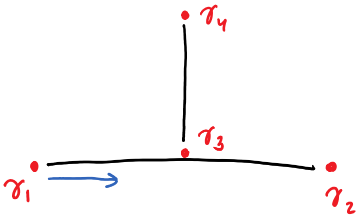
γ_2 collides with the branch cut of γ_2 once.

How to "braid" Majoranas on edges of wires ?

(Alicea, Oreg, von Oppen, Refael, Fisher 10')

"exchange" γ_1, γ_2 ?

T - junction geometry: (Two "auxiliary" Majoranas γ_3, γ_4:)



Change position of γ_i (e.g., by applying gate potentials):

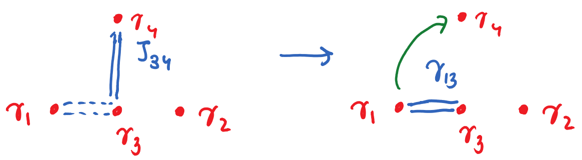
When γ_3 is close to γ_4, $H_{eff} = i\,J_{34}(r_{34})\,\gamma_1\gamma_4$.

More generally $H_{eff} = \sum_{i,j} i\, J_{ij}(t)\, \gamma_i \gamma_j$

Make the J_{ij}'s time dependent!

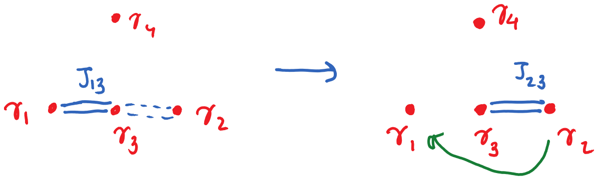
"Discrete Braiding protocol":

Step I :

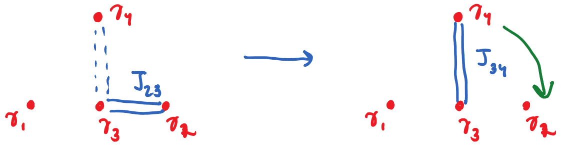


(During this step, the g.s. degeneracy does not change!)

Step II:



Step III :



H returned to itself. However, the wavefunction does not.

E.g., in step I : $H_\lambda = -J(1-\lambda)\, i\gamma_3\gamma_4 - J\lambda\, i\gamma_1\gamma_3$

(γ_2 never participates)

λ goes from 0 to 1 adiabatically

Find zero mode of H_λ : $\gamma(\lambda) = \sum_{j=1,3,4} a_j \gamma_j$

require $[\gamma(\lambda), H_\lambda] = 0$

$[\gamma_1, \gamma_1\gamma_3] = 2\gamma_3$ etc.

$$[\gamma(\lambda), H] = a_1\left(-2\lambda J\, i\, \gamma_3\right) + a_3\left(-2(1-\lambda) J \gamma_4 + 2\lambda J \gamma_1\right)$$
$$+ a_4\left(+2(1-\lambda) J\, i\, \gamma_3\right) = 0$$

$$\begin{cases} -\lambda a_1 + (1-\lambda) a_4 = 0 \\ a_3 = 0 \end{cases}$$
$$\Downarrow$$

$\lambda = 0 : \; a_4 = 0, \quad a_1 = 1$

$\lambda = 1 : \; a_1 = 0, \quad a_4 = +1$ (you can find the sign by requiring that $a_4(\lambda)$ is continuous.)

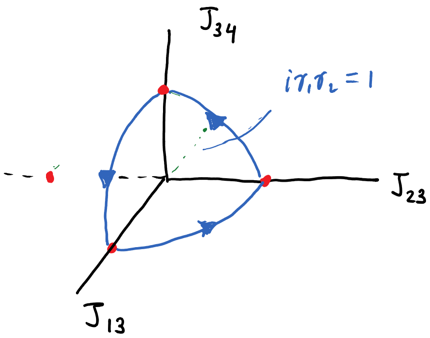
One can follow the procedure through steps II, III, and find that at the end $\begin{cases} \gamma_1 \to \pm \gamma_2 \\ \gamma_2 \to \mp \gamma_1. \end{cases}$ as anticipated.

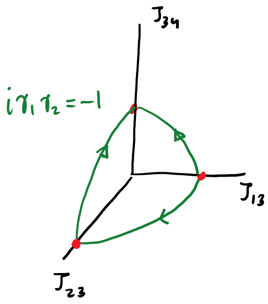
What we have done here is drawn a closed loop in Hamiltonian space, rather than real space.

The braiding protocol can be visualized as follows.

The Hamiltonian has 3 parameters, J_{13}, J_{23}, and J_{34}.

Draw them in a 3D space and fix $J_{13}^2 + J_{23}^2 + J_{34}^2 = J^2 =$ co





At any given intermediate time, there are only two of the three parameters which are non-zero. The trajectory cove $\frac{1}{8}$ of the area of the sphere; For a spin $\frac{1}{2}$ (2-le system), that gives a Berry phase of $\phi = \frac{1}{2} \cdot 4\pi \cdot \frac{1}{8} =$

e.g. it is spanned by $i\gamma_1\gamma_2 = \pm 1$; you can check that on this basis the following representation is possible

$$i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$$

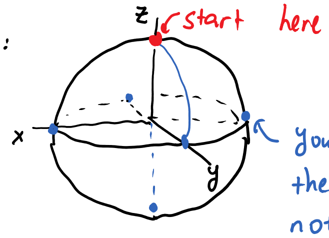
$$i\gamma_1\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x$$

$$i\gamma_1\gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^y$$

I.e. these form a "Pauli group".

The group is preserved by the braiding (meaning that every element is mapped to another element, but never a sum of two).

The braiding can be thought of as rotations by $\frac{\pi}{2}$ around x, y, z:



you can reach these points, but nothing else.

This result generalizes to an arbitrary number of qbits. One can show that if the braiding preserves the a Pauli group defined on an n-qbit Hilbert space, then it is not universal.

Simpler strategy to “demonstrate” the non-Abelian nature, without full braiding:

create from the vaccum two pairs of zero modes, $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \gamma_4\}$, such that initially $i\gamma_1\gamma_2 = i\gamma_3\gamma_4 = +1$. Now, what is the representation of $i\gamma_2\gamma_3$ in this basis?

$$i\gamma_2\gamma_3 = i\frac{f_1 - f_1^\dagger}{i}\left(f_2 + f_2^\dagger\right). \tag{21}$$

In the subspace of total even parity, $\{|0,0\rangle, |1,1\rangle\}$, the matrix elements of this operator are:

$$\langle 00|i\gamma_2\gamma_3|11\rangle = 1$$

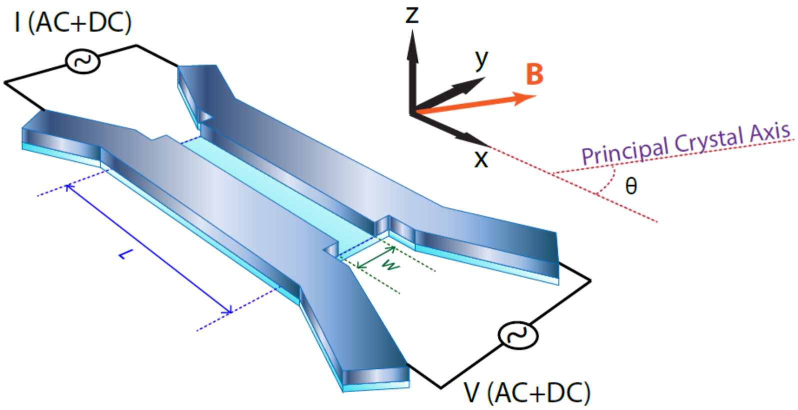
I.e., in this basis, where $i\gamma_1\gamma_2$ acts as σ^z, $i\gamma_2\gamma_3$ acts as σ^x. (We can also check that $i\gamma_1\gamma_3 = i\left(f_1 + f_1^\dagger\right)\left(f_3 + f_3^\dagger\right)$ acts as $-\sigma^y$.)

Therefore, if we start from $i\gamma_1\gamma_2 = i\gamma_3\gamma_4 = +1$ and then “fuse” γ_2 and γ_3 together, we get either $i\gamma_2\gamma_3 = \pm 1$ with a probability of 50%, independently of how exactly the process is implemented.

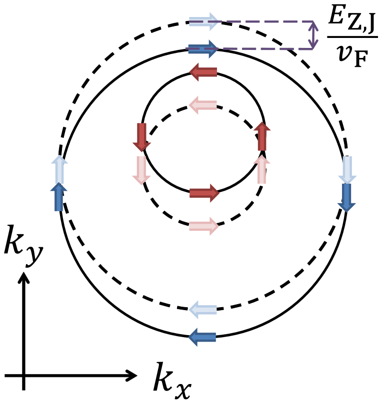
VI. NOVEL PLATFORMS IN 2D SYSTEMS

F. Pientka, A. Keselman, EB, Ady Stern, A. Yacoby, B. Halperin, Phys. Rev. X (2017).

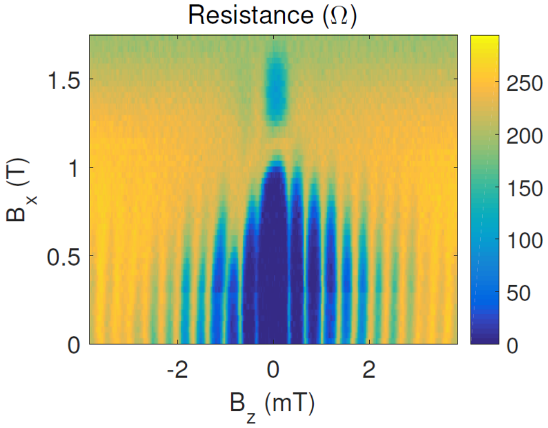
Hart et al. experiment (Yacoby group):



Shifted Fermi surfaces due to Zeeman field:



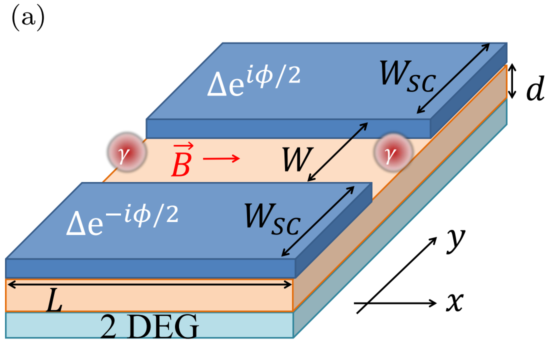
As a function of B_x, the critical current vanishes and recovers:



All the ingredients are there to realize topological superconductivity in the "wire" (the segment of the semiconductor between the superconductors): spin-orbit coupling, broken time reversal, superconductivity!

What is the phase diagram as a function of the phase difference ϕ between the superconductors, and the magnetic field?

Topological phase transition has to be accompanied with a closing of the gap (level crossing) at $k_x = 0$.



Examine the spectum at $k_x = 0$. Consider the Hamiltonian

$$H = \left(\begin{array}{cccc} c^\dagger_{k\uparrow} & c^\dagger_{k\downarrow} & c_{-k\downarrow} & -c_{-k\uparrow} \end{array} \right) \left(\begin{array}{cc} \frac{k_x^2}{2m} - \alpha k_x \sigma^y - \mu - B(x)\sigma^y & \Delta(x) \\ \Delta^*(x) & -(\frac{k_x^2}{2m} - \alpha k_x \sigma^y - \mu) - B(x)\sigma^y \end{array} \right) \left(\begin{array}{c} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\downarrow} \\ -c^\dagger_{-k\uparrow} \end{array} \right)$$

where

$$\Delta(x) = \Theta(x - w/2)e^{i\phi/2} + \Theta(-x - w/2)e^{-i\phi/2}$$

and

$$B(x) = B[\Theta(w/2 - x) - \Theta(-w/2 - x)].$$

First, set $B = 0$. Then, in the limit where $\mu \gg \Delta$, we get a level crossing at zero energy when $E = 0$. This is *not* a topological transition, since it occurs simultaneously for $\sigma^y = \pm 1$.

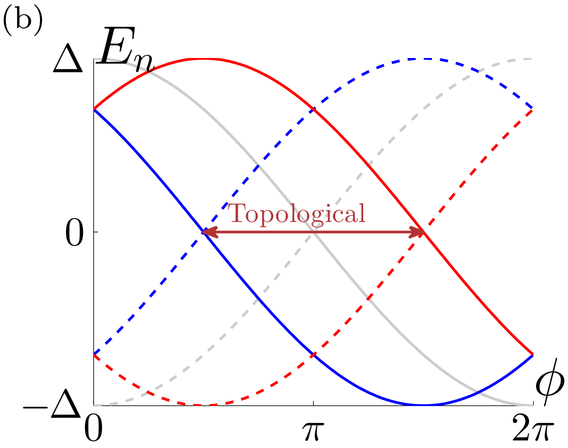
To see that thereis a zero-energy state at $\phi = \pi$, linearize the dispersion around $k = k_F$. Moreover, since normal reflection is suppressed, right moving electrons are only coupled to left moving holes. The Hamiltonian takes the form

$$H = \left(\begin{array}{cc} \psi^\dagger_R & \psi_L \end{array} \right) \left(\begin{array}{cc} -iv_F\partial_x & \Delta(x) \\ \Delta^*(x) & iv_F\partial_x \end{array} \right) \left(\begin{array}{c} \psi_R \\ \psi^\dagger_L \end{array} \right)$$

Then, as we saw before, we get a zero energy state if the signs of $\Delta(x \to \infty)$ and $\Delta(x \to -\infty)$ are opposite. More generally, there is are in-gap Andreev bound states whose energy goes as

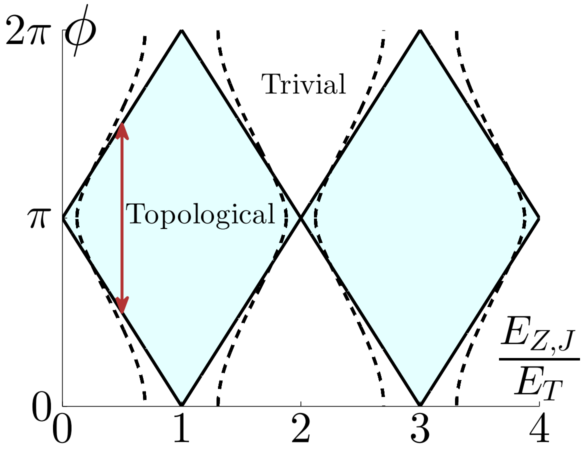
$$E = \pm\Delta\cos(\phi/2).$$

Now, applying a Zeeman field shifts the spin up and down levels oppositely:

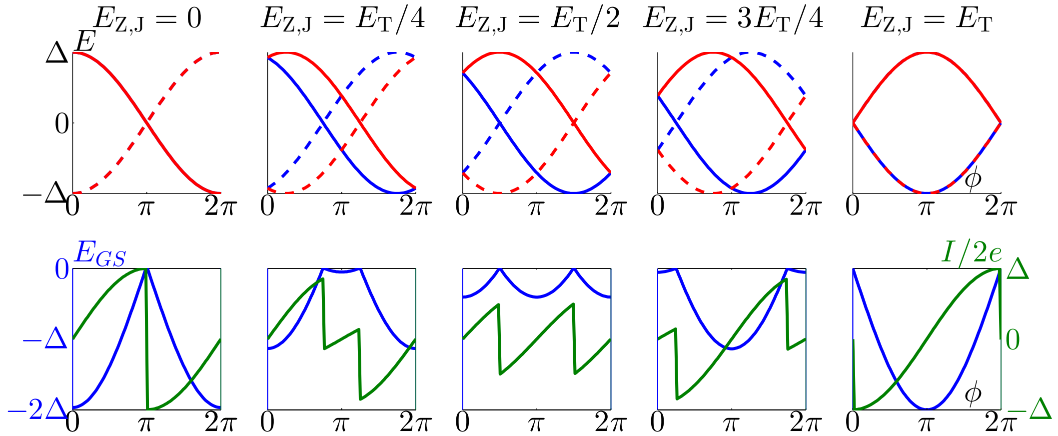


The region that opens between the two level crossings has to be topological! (*Note:* This is independent of the chemical potential.)

(ϕ, B) phase diagram:



If the system is not phase biased, as we saw, the critical current undergues a minimum at a certain magnetic field, $B \sim \frac{\hbar v_F}{\mu_B g w}$. What happens at that field?



This is a first-order quantum phase transition from a state with $\phi \approx 0$ to a state with $\phi \approx \pi$; the latter is the topological state, that has Marjoana zero modes at its ends.

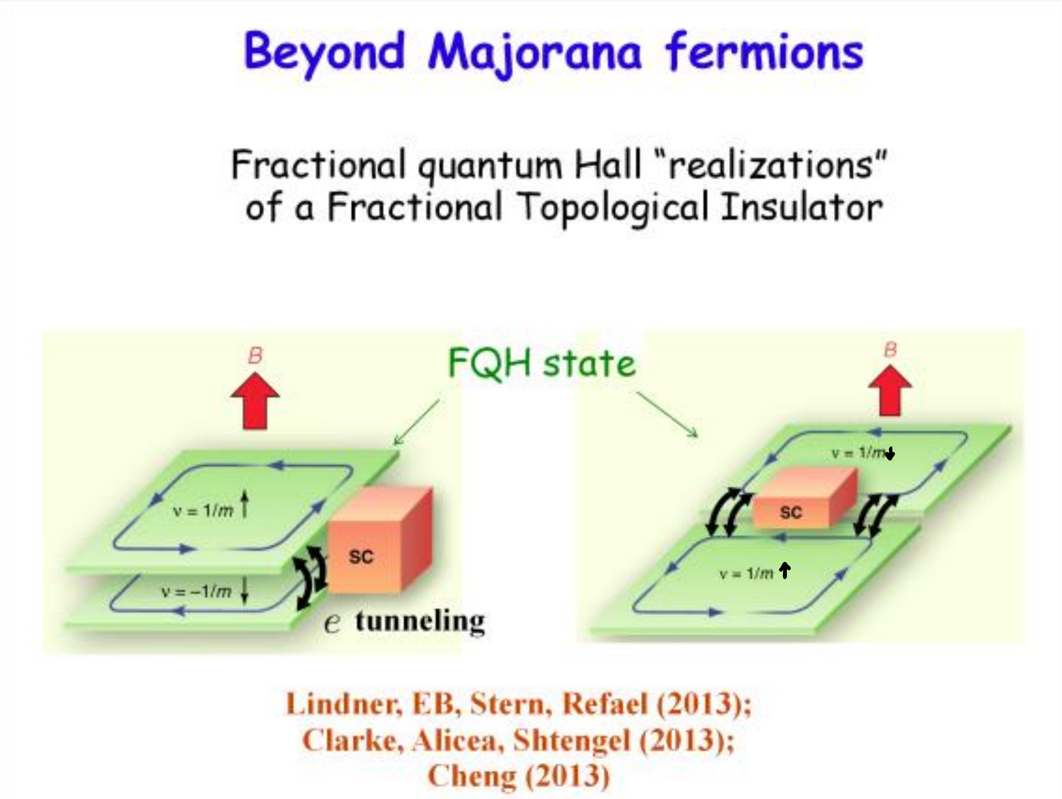
VII. BEYOND MAJORANAS

The outstanding question is: how to go beyond Majorana fermions? I.e., how to get non-Abelian topological states that have richr braiding properties, possibly even universal?

One strategy is to hope to get lucky. However, nature has not been extremely generous in providing topologically ordered states. There is now good evidence that such a state occurs in the quantum Hall platea at $\nu = 5/2$ (in particular, the thermal conductance is $\kappa/T \approx \frac{1}{2}\pi^2 k_B^2/3h$, "half" of the natural value.) But even if this is confirmed, it is still the Moore-Read state, that does not support non-Abelian statistics beyond that of Majoranas.

Another way is to take existing ingredients and two to engineer them in new ways. (This approach has fruitful in the Majorana case!). It turns out that, quite generally, topologically ordered Abelian phases can support "defects" (such as special kinds of domain walls, lattice dislocations, etc - see examples below) that support non-Abelian statistics. This non-Abelian statistics can “enrich” the behavior of the original phase.

One example is the case of the edge of a “fractional topological insulator”, or equivalently, a “trench” in a fractional quantum Hall state.



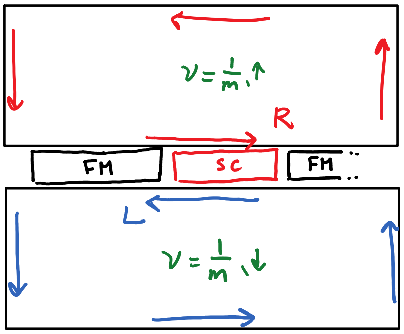
Beyond Majorana fermions

Fractional quantum Hall “realizations”
of a Fractional Topological Insulator

Lindner, EB, Stern, Refael (2013);
Clarke, Alicea, Shtengel (2013);
Cheng (2013)

There are three phases of the edge (thought of as a one-dimensional system): gapless, a phase dominated by normal tunneling, and a phase dominated by Cooper pair tunneling due to the proximity to the superconductor.

What happens at the interface between the two gapped phases?



(m = odd integer)

New type of zero mode: “Fractionalized Majoranas” or “parafermions”!

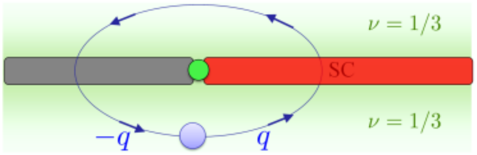

Parafarmacia
APERTO
RACCOLTA
FARMACI
SCADUTI

In the superconduting regions on the edge, a pair of quasi-particles from the bulk can be added with zero energy cost.

In the normal regions, a quasi-particle-quasi hole pair can be added with zero energy cost.

Fractionalized zero modes at "twist defects" in topological phases

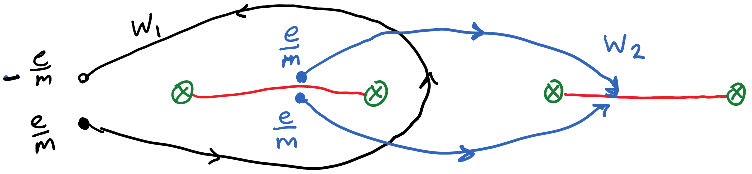
Ends of line defects that interchange anyon types ("topological symmetry")



The "defect line" can permute anyon types.

This non-contractable loop is describe by an operator acting on the zero-energy subspace. Moreover, operators corresponding to different non-contractable loops do not commute:

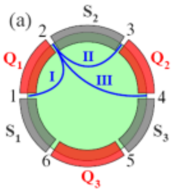
Relation to quantum dimension:



$W_1 W_2 = e^{-\frac{4i\pi}{m}} W_2 W_1$ => G.S. is at least m-fold degenerate!

This mandates a ground state degeneracy of at least m per superconducting domain. I.e., each SC-normal interface carries a "zero mode" with quantum dimension $\sqrt{m}$. A more careful analysis shows that there is also a Marjoana zero mode at the interface; the true quantum dimension is, in fact, $\sqrt{2m}$.

A braiding operation of interfaces can be defined, similarly to the Majorana case.



$$U_{34} = \exp\left(i\frac{\pi m}{2}\hat{Q}_2^2\right) = \exp\left(i\frac{\pi}{2m}q_2^2\right)$$

$$Q_2 = \frac{1}{m}q_2, \qquad q_2 = 0, \ldots, 2m-1$$

Example: m=3 $\quad q_2 = 2p + 3q \quad (p = 0, 1, 2, \ q = 0, 1)$

$$U_{34} = \exp\left(i\frac{\pi}{6}q_2^2\right) = \exp\left(-i\frac{\pi}{2}q^2\right)\exp\left(i\frac{2\pi}{3}p^2\right)$$

(Majorana) ⊗ (Something new!)

This operation turns out to be non-universal, as in the Majorana case. However, defects enrich the properties of the underlying topologically ordered phase. In the case above, we started from an Abelian phase and ended with a non-Abelian one. In the case of a phase which is already non-Abelian, the properties are enriched further. For example, there are cases where one starts with a non-Abelian but non-universal phase, and ends with a universal one.