University of Basel

Classical and quantum synchronization

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QSIT Quantum Science and Technology

National Centre of Competence in Research

synchronization: synchronized events

different "agents" act synchronously, at the same time:

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- audience leaves for the coffee break

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- rowing (e.g. coxed eight)

synchronization: synchronized events

different "agents" act synchronously, at the same time:

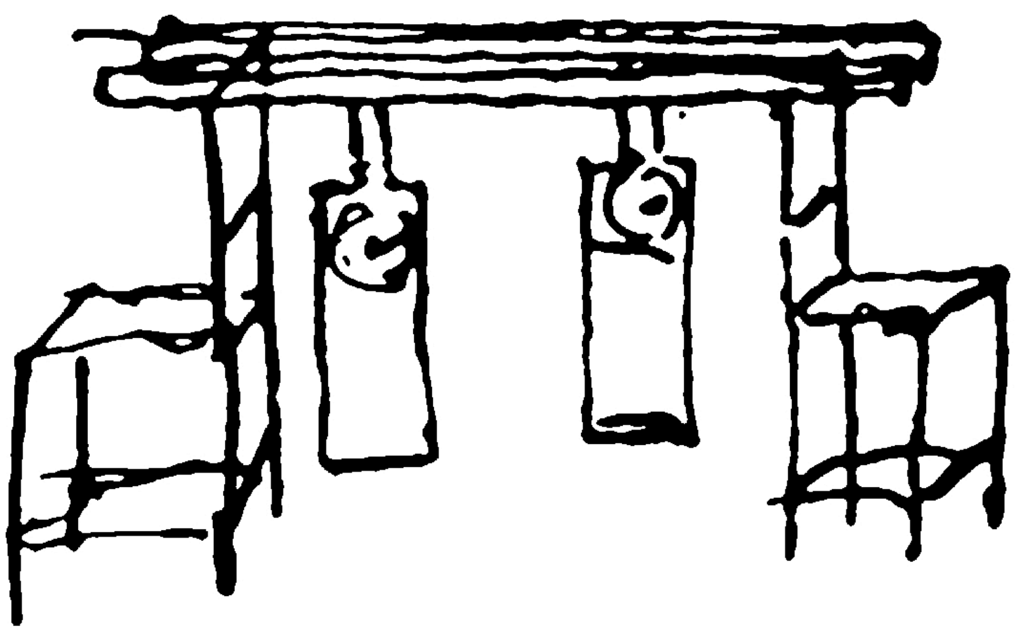
- audience leaves for the coffee break
- rowing (e.g. coxed eight)

control by external "clock"

spontaneous synchronization

spontaneous synchronization

Huygens' observation (1665):
two pendulum clocks fastened to the same beam will synchronize (anti-phase)



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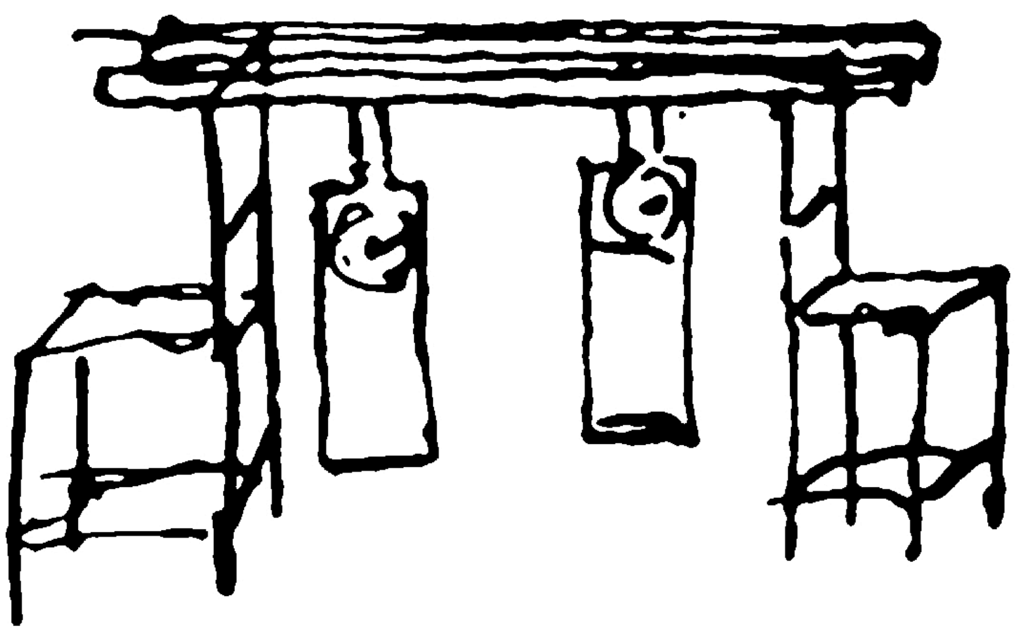

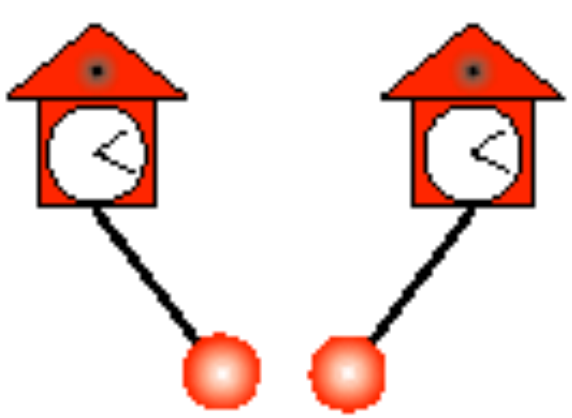

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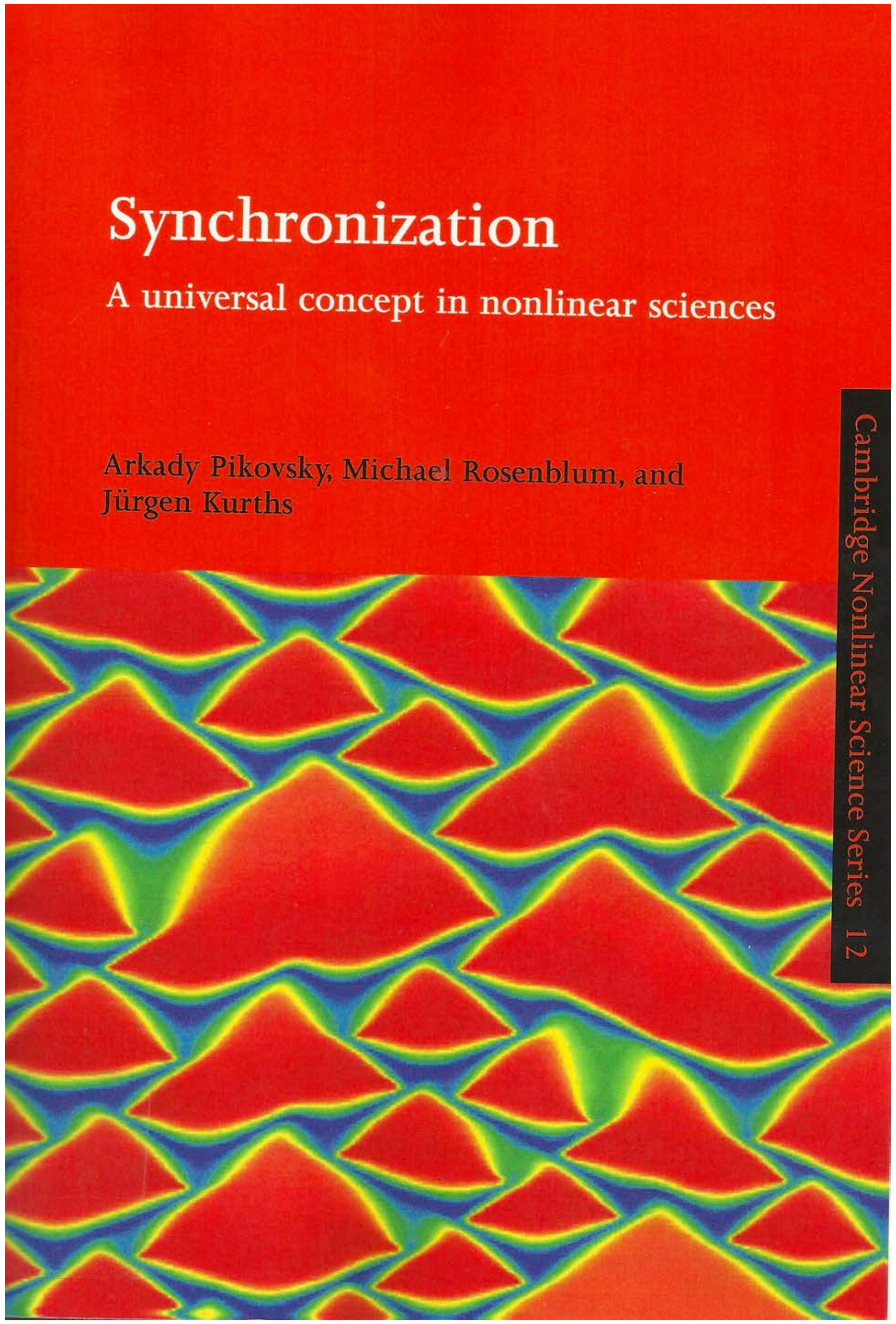
spontaneous synchronization

nice introduction to classical synchronization



spontaneous synchronization

- rhythmic applause in a large audience
- heart beat (due to synchronization of 1000's of cells)
- synchronous flashing of fireflies

global outline

- Lecture I: classical synchronization
- Lecture II: quantum synchronization
- Lecture III: topics in quantum synchronization

lecture I: classical synchronization

- synchronization of a self-oscillator by external forcing
- two coupled oscillators
- ensembles of oscillators: Kuramoto model
- realization in a one-dimensional Josephson array

definition of self-oscillator

self-oscillator or self-sustained (limit-cycle) oscillator

- driven into oscillation by some energy source
- maintains stable oscillatory motion when unperturbed or weakly perturbed
- intrinsic natural frequency ω_0

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examples:
(i) pendulum clock
(ii) van der Pol oscillator $\ddot{x} + (-\gamma_1 + \gamma_2 x^2)\dot{x} + \omega_0^2 x = 0$

synchronization problem

given two (or more) self-oscillators with (slightly) different frequencies: $\omega_0, \omega_0', \ldots$
will they agree on ONE frequency if coupled?

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let's start with an easier problem:

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= synchronization by external forcing

linear oscillators

NOTE:
driven linear damped harmonic oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \Omega \cos(\omega_d t)$$

solution: damped eigenmodes + $A \cos(\omega_d t)$

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the same applies to eigenmodes of coupled linear harmonic oscillators

non-linearity is crucial for synchronization

synchronization by external forcing

phase $\phi(t)$ parametrizes motion along one cycle of the oscillator
amplitude is assumed to be constant

undisturbed dynamics: $$\frac{d\phi(t)}{dt} = \dot{\phi}(t) = \omega_0$$

synchronization by external forcing

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undisturbed dynamics: $$\frac{d\phi(t)}{dt} = \dot{\phi}(t) = \omega_0$$

always possible by re-parametrization:

$\tilde{\phi}(t)$ non-uniform $\Rightarrow$

$$\phi(t) = \omega_0 \int_0^{\tilde{\phi}} d\tilde{\phi} \left(\frac{d\tilde{\phi}}{dt} \right)^{-1}$$ uniform

synchronization by external forcing

drive by a periodic force of frequency $\omega_d \approx \omega_0$ and amplitude ϵ :

$$\dot{\phi}(t) = \omega_0 + \epsilon Q(\phi, \omega_d t)$$

where Q is 2π-periodic in both arguments

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define deviation $\Delta\phi = \phi - \omega_d t$

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Fourier-expansion of Q and averaging
(→ vanishing of rapidly oscillating terms) leads to

$$\frac{d\Delta\phi(t)}{dt} = \omega_0 - \omega_d + \epsilon q(\Delta\phi)$$ Adler 1946

synchronization by external forcing

$$\frac{d\Delta\phi(t)}{dt} = \omega_0 - \omega_d + \epsilon q(\Delta\phi)$$

q is 2π -periodic

simplest choice: $q(\Delta\phi) = \sin(\Delta\phi)$

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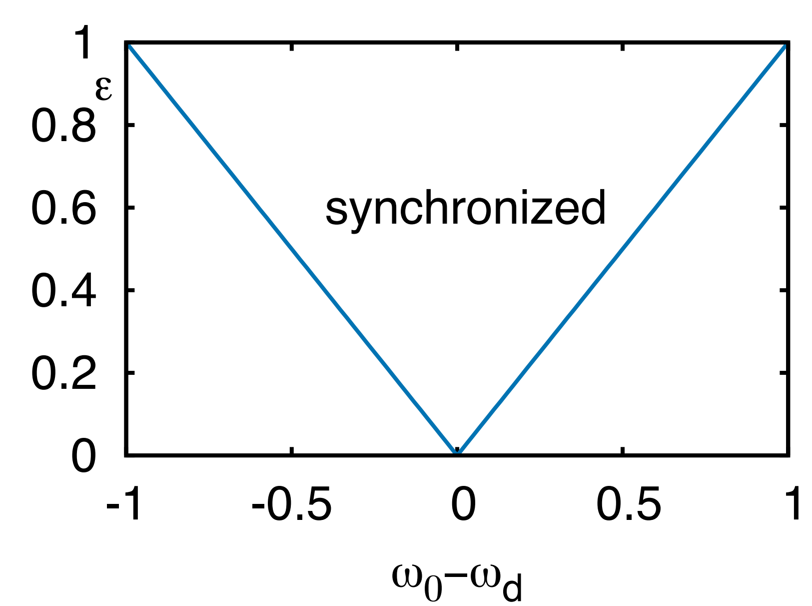
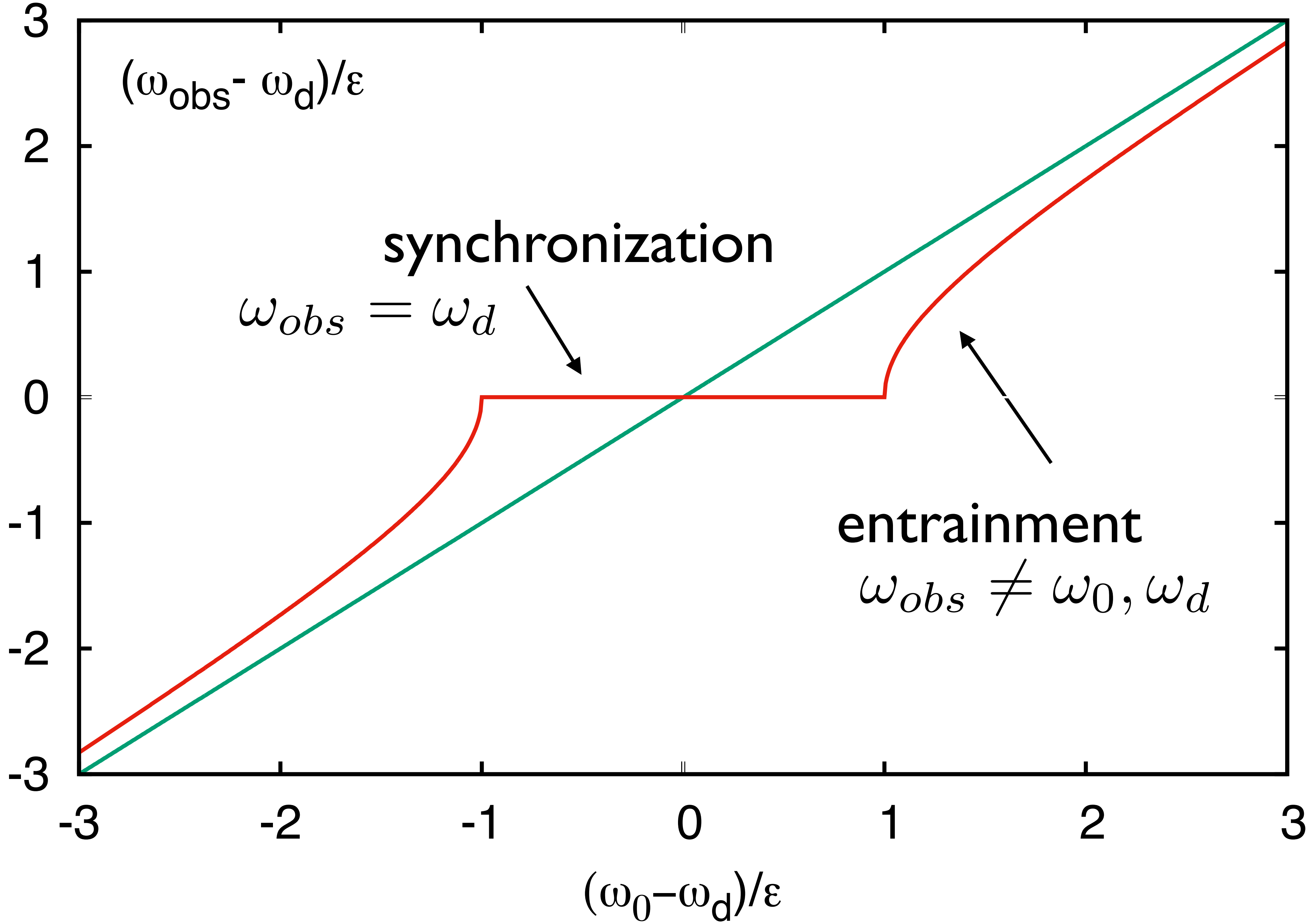
synchronization, i.e., $\frac{d\Delta\phi(t)}{dt} = 0$

possible for $|\frac{\epsilon}{\omega_0 - \omega_d}| > 1$



Adler plot of observed frequency

$\omega_{obs} = \langle \frac{d\phi}{dt} \rangle$ (time average); differs in general from both ω_0 and ω_d



side remark

Adler equation $$\frac{d\Delta\phi(t)}{dt} = \omega_0 - \omega_d + \epsilon \sin(\Delta\phi)$$

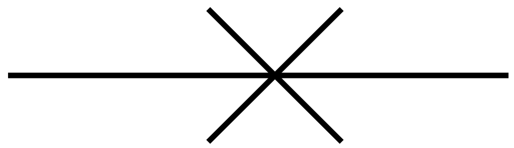
appears in many areas of physics: e.g.

superconductivity (Shapiro steps)

quantum optics (ring-laser gyros)

current-biased Josephson junction

Josephson relations $I = I_c \sin\phi$ $$\frac{d\phi}{dt} = \frac{2e}{\hbar}V$$



E_J

$$I_c = \frac{2e}{\hbar}E_J$$

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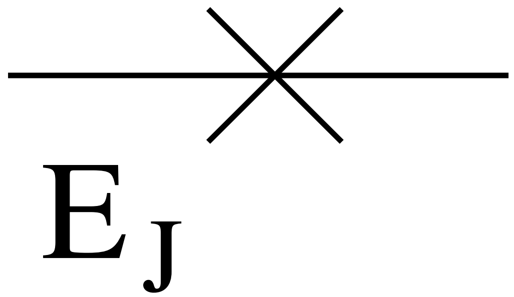
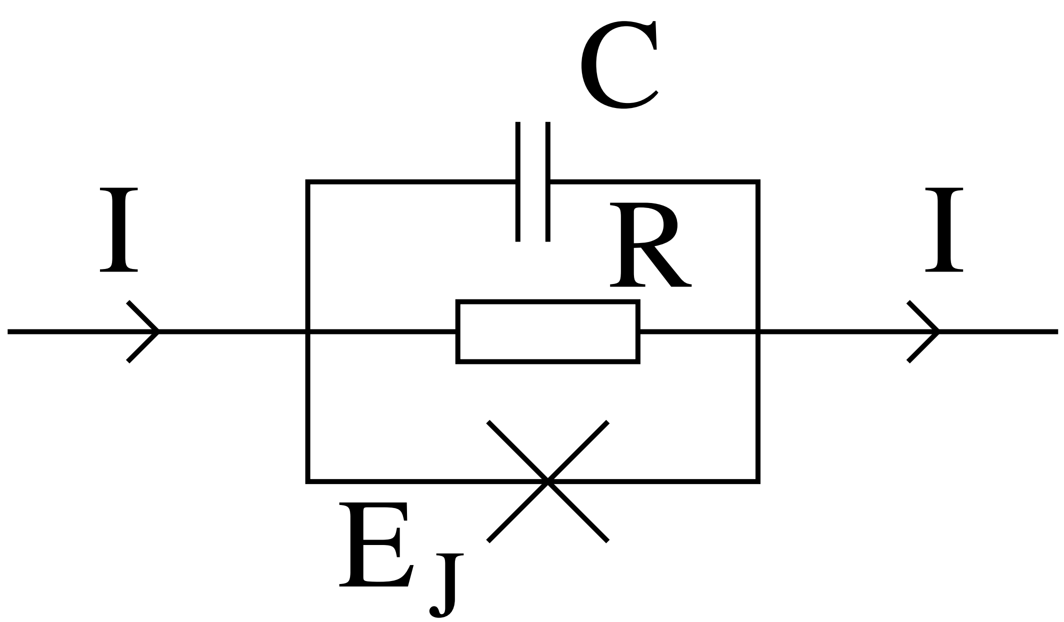


RSJ model



$$I = I_c \sin\phi + \frac{\hbar}{2eR}\frac{d\phi}{dt} + \frac{C\hbar}{2e}\frac{d^2\phi}{dt^2}$$

current-biased Josephson junction

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classical junction: $C \to 0$

$$\frac{d\phi}{dt} = \frac{2e}{\hbar}RI - \frac{2e}{\hbar}RI_c \sin\phi$$

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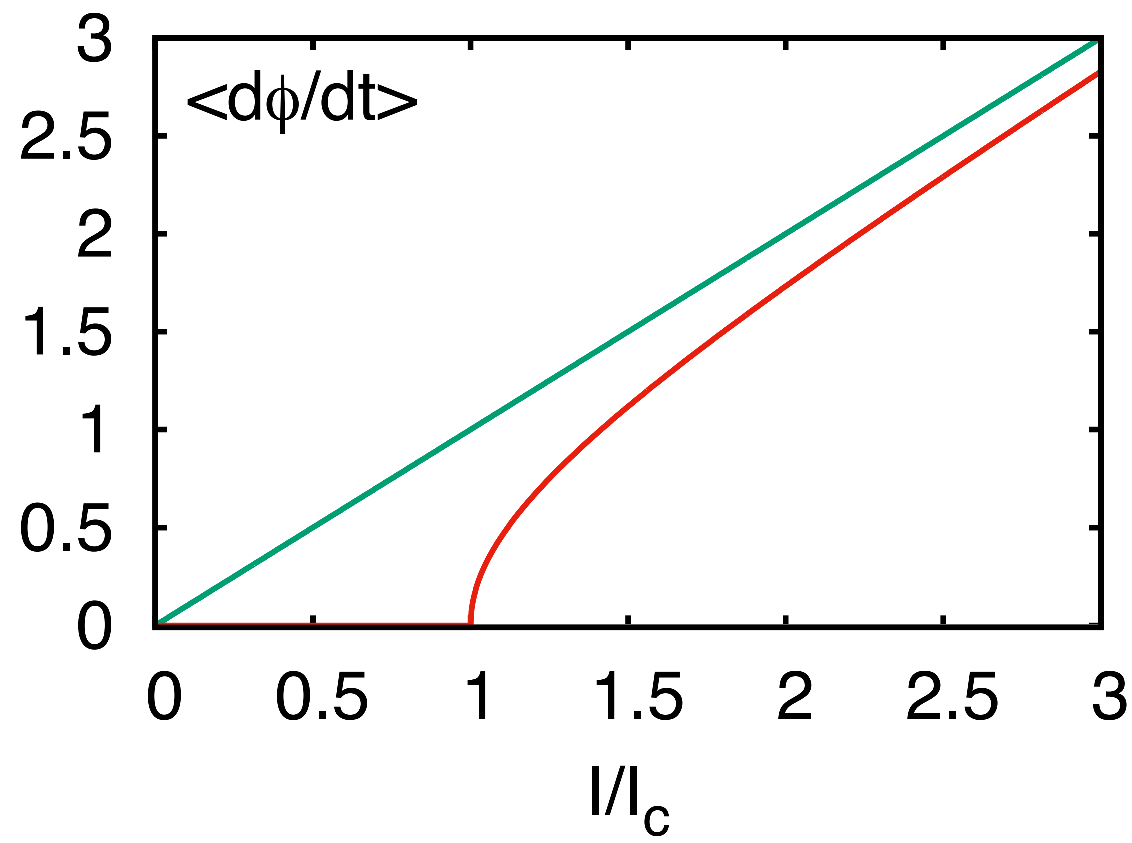


"synchronized" state with

$$\langle\frac{d\phi}{dt}\rangle = \frac{2e}{\hbar}\langle V\rangle = 0$$

for "detuning" $I/I_c < 1$

current-biased Josephson junction

- strictly speaking, J junction is a rotator
  (and not a self-sustained oscillator)

current-biased Josephson junction

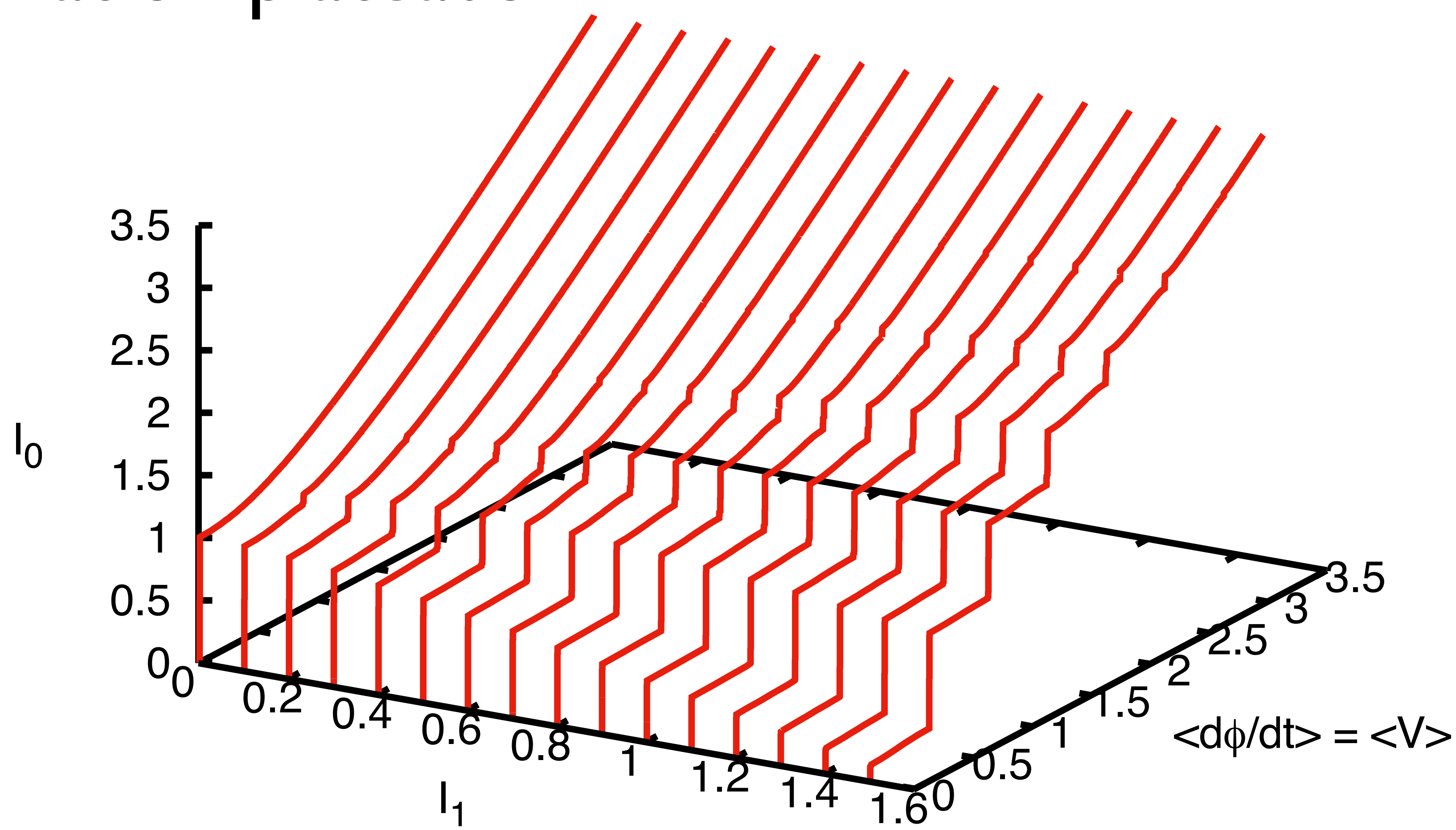
- strictly speaking, J junction is a rotator (and not a self-sustained oscillator)
- can be synchronized by a periodic external force (Shapiro steps) or to another J junction

Shapiro steps

current-biased Josephson junction + harmonic current bias:

$$I_0 + I_1 \sin \omega_d t = \frac{\hbar}{2eR}\frac{d\phi}{dt} + I_c \sin \phi$$

additional synchronization plateaus



2 oscillators, no external forcing

undistorted frequencies ω_1 , ω_2

weak interaction affects only the phases ϕ_1 , ϕ_2

$$\dot{\phi}_1(t) = \omega_1 + \epsilon Q_1(\phi_1, \phi_2)$$
$$\dot{\phi}_2(t) = \omega_2 + \epsilon Q_2(\phi_2, \phi_1)$$

Fourier expansion, averaging to get rid of rapidly oscillating terms leads to

$$\frac{d\Delta\phi(t)}{dt} = \omega_1 - \omega_2 + \epsilon q(\Delta\phi)$$

Adler equation!

2 oscillators, no external forcing

the two oscillators will lock in on a common frequency between ω_1 and ω_2

→ spontaneous synchronization

ensembles of coupled oscillators

N coupled phase oscillators $\phi_i(t)$,
random frequencies ω_i described by probability density $g(\omega)$

$$\dot{\phi}_i = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\phi_j - \phi_i), \qquad i = 1, \ldots, N$$

Kuramoto model (1975),
solvable for infinite-range coupling $K_{ij} = \epsilon / N > 0$

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non-equilibrium phase transition to a synchronized state
as a function of ϵ

ensembles of coupled oscillators

degree of synchronicity described by order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}$$

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$\psi(t)$ is the average phase

transition to $r \neq 0$ → synchronization

$0 \leq r(t) \leq 1$ measures the coherence of the ensemble

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substituted back in the Kuramoto equation gives

$$\dot{\phi}_i = \omega_i + \epsilon r \sin(\psi - \phi_i), \qquad i = 1, \ldots, N$$

→each oscillator couples to the common average phase $\psi(t)$

partial coherence

$$\dot{\phi}_i = \omega_i + \epsilon r \sin(\psi - \phi_i), \qquad i = 1, \ldots, N$$

interpretation of $0 < r < 1$:

typical oscillator running with velocity $\omega - \epsilon r \sin(\phi - \psi)$ will become stably locked at an angle such that

$$\epsilon r \sin(\phi - \psi) = \omega \qquad -\pi/2 \leq \phi - \psi \leq \pi/2$$

oscillators with frequencies $|\omega| > \epsilon r$ cannot be locked

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→ three groups:
(i) synchronized
(ii) unsynchronized, velocity $> \dot{\psi}$
(iii) unsynchronized, velocity $< \dot{\psi}$

Kuramoto's results

$g(\omega) = g(-\omega)$

J.A. Acebron et al., Rev. Mod. Phys. 77, 317 (2005)

for $N \to \infty$, transition from incoherent state $(r = 0)$ to partially coherent $(r > 0)$ state occurs at

$$\epsilon_c = \frac{2}{\pi g(\omega = 0)}$$

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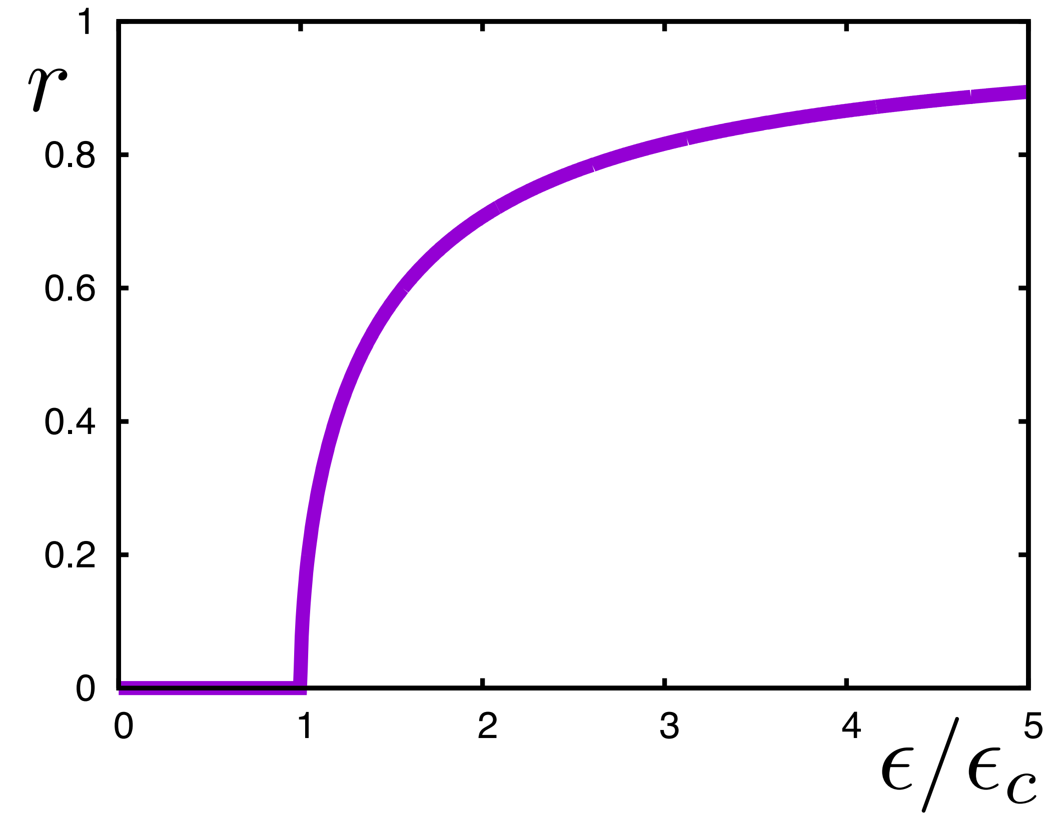

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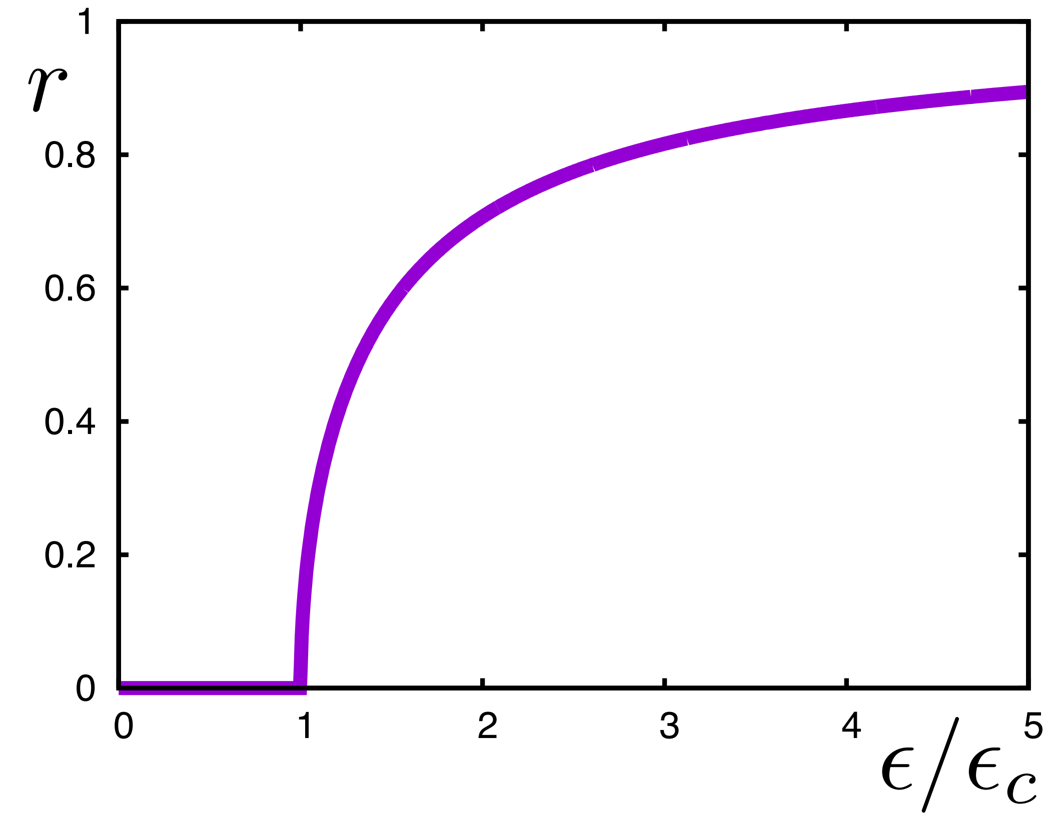

$$g(\omega) = \frac{1}{\pi}\frac{\gamma}{\gamma^2 + \omega^2} \qquad r = \sqrt{1 - \frac{\epsilon_c}{\epsilon}} \quad \text{for} \quad \epsilon > \epsilon_c = 2\gamma$$

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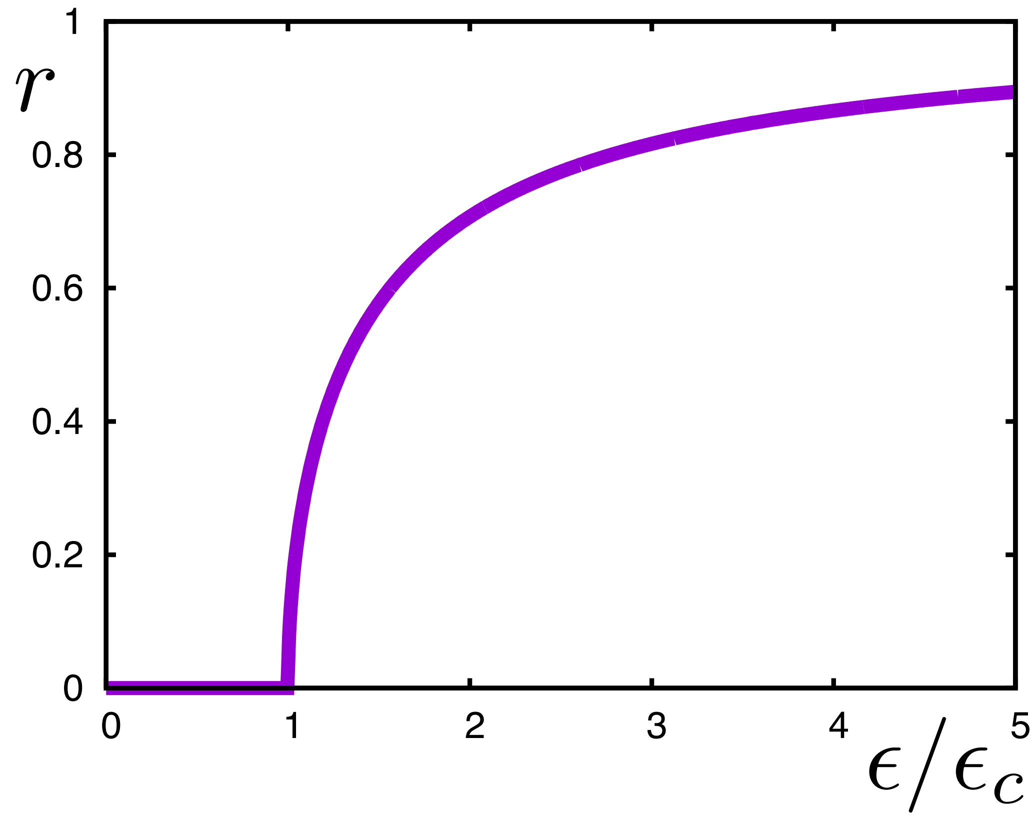

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exact solution - pretty amazing!

simulation with N=900 oscillators

example: $\gamma = 1$ $\qquad$ $g(\omega) = \frac{1}{\pi}\frac{1}{1+\omega^2}$

synchronization transition at $\epsilon_c = 2$

$r = \sqrt{1 - \frac{\epsilon_c}{\epsilon}} \approx 0.71$ for $\epsilon = 4$

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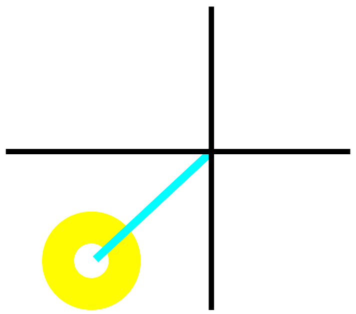
nice simulation program: Synched by Per Sebastian Skardal

“K” corresponds to ϵ



gives amplitude and phase of the order parameter

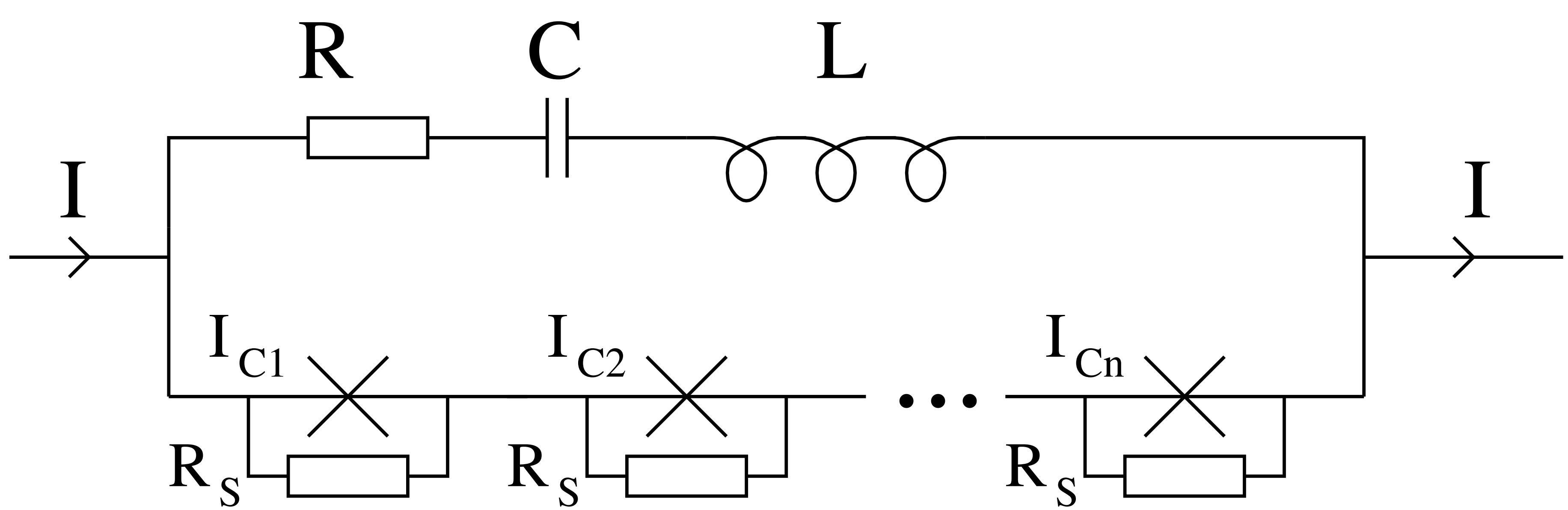
https://sites.google.com/site/persebastianskardal/software/synched

realization in a one-dim Josephson array

K. Wiesenfeld, P. Colet, and S.H. Strogatz, PRL 76, 404 (1996)

uncoupled Josephson junctions
= rotators with $\omega_i = \frac{2e}{\hbar} R_S \sqrt{I^2 - I_{Ci}^2}$

global coupling by RCL branch
⇒ natural realization of the Kuramoto model



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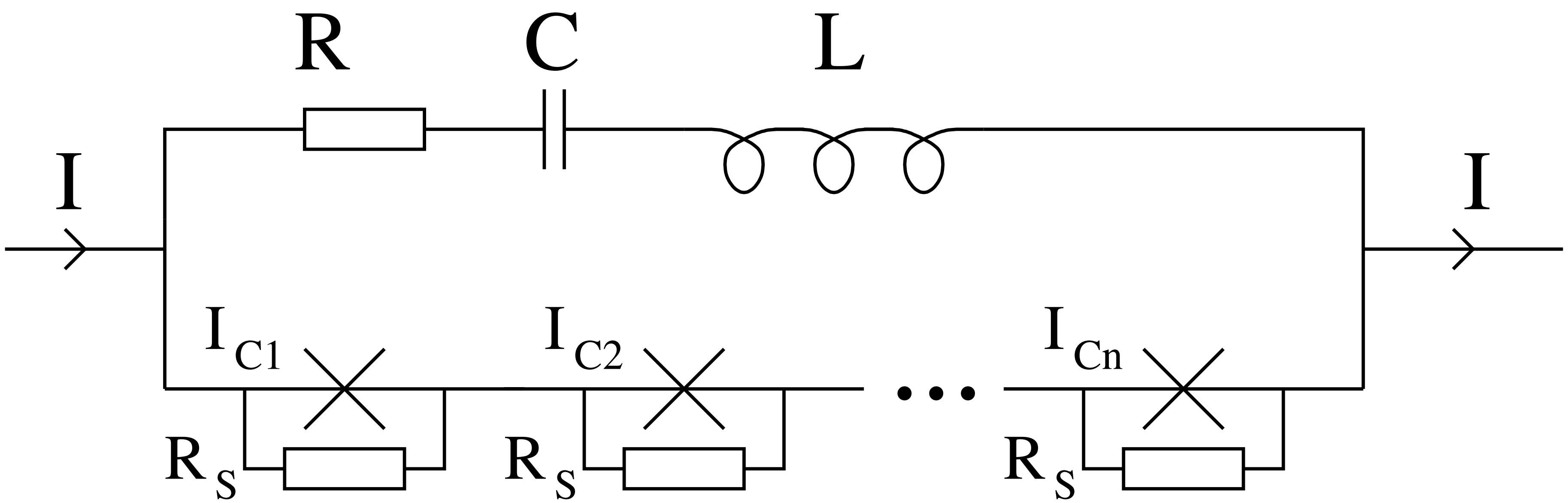

$$I - \dot{Q} = I_{Ck} \sin \phi_k + \frac{\hbar}{2eR_S} \dot{\phi}_k$$ for each junction

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \frac{\hbar}{2e} \sum_k^N \dot{\phi}_k$$

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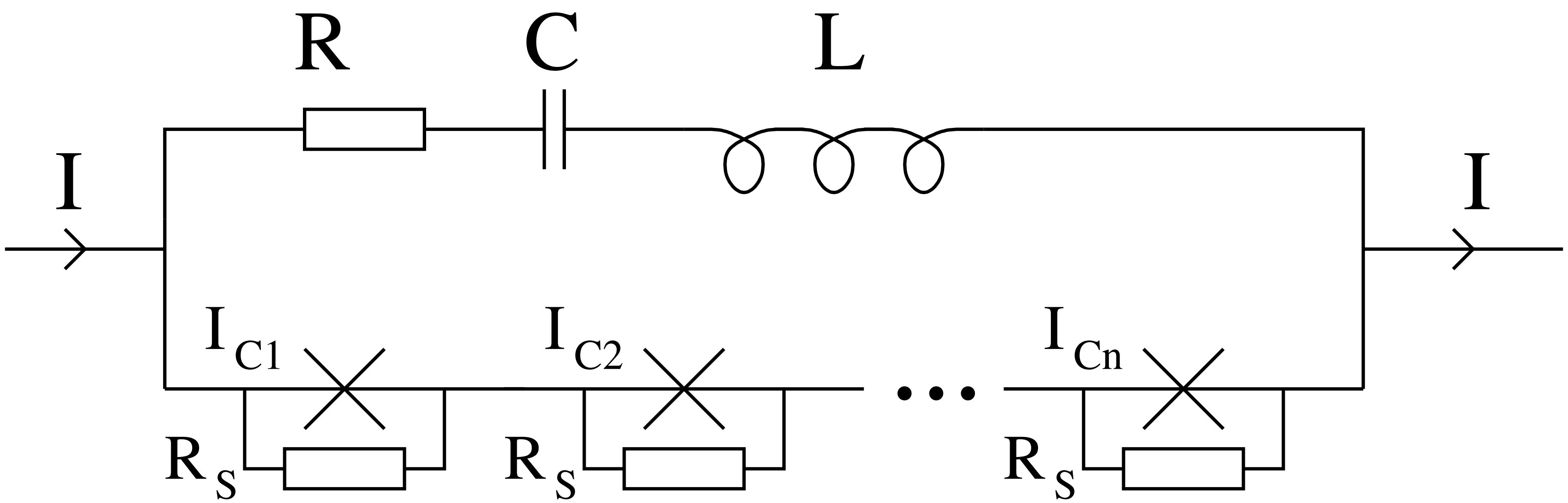

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uniformly rotating phases θ_k in the uncoupled case $\dot{Q} = 0$

$$\frac{d\theta_k}{\omega_k} = dt = \frac{d\phi_k}{(2eR_S/\hbar)(I - I_{Ck} \sin \phi_k)}$$

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combined with $I - \dot{Q} = I_{Ck} \sin \phi_k + \frac{\hbar}{2eR_S} \dot{\phi}_k$ yields

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first-order averaging ⇒

$$\dot{\theta}_k = \omega_k - \frac{K}{N} \sum_{j}^{N} \sin(\theta_k - \theta_j + \alpha)$$ Kuramoto!

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$$K = \frac{NR_S\bar{\omega}(2eR_S I/\hbar - \bar{\omega})}{[(L\bar{\omega}^2 - 1/C)^2 + \bar{\omega}^2(R + NR_S)^2]^{1/2}}$$

$$\cos\alpha = \frac{L\bar{\omega}^2 - 1/C}{[(L\bar{\omega}^2 - 1/C)^2 + \bar{\omega}^2(R + NR_S)^2]^{1/2}}$$

quantum synchronization

so far only **classical** non-linear systems

synchronization in **quantum** systems:

- experimental situation?
- does it exist at all?
- how to quantify and measure it?
- relation to other measures of `quantumness' (entanglement, mutual information, ...)

conclusion

- classical synchronization is well-studied,
- simplest model: one self-oscillator + external forcing → Adler equation

$$\frac{d\Delta\phi(t)}{dt} = \omega_0 - \omega_d + \epsilon \sin(\Delta\phi)$$

- frequency locking if detuning < drive strength
- two oscillators lock if detuning < coupling
- synchronization (phase) transition in ensembles of mutually coupled self-oscillators: Kuramoto model

appendix