University of Basel

Classical and quantum synchronization

Christoph Bruder - University of Basel

Basel Center for Quantum Computing & Quantum Coherence

QSIT Quantum Science and Technology

National Centre of Competence in Research

global outline

- Lecture I: classical synchronization
- Lecture II: quantum synchronization
- Lecture III: topics in quantum synchronization

lecture II: quantum synchronization

- experimental examples: synchronization in microsystems
- toy model: quantum van der Pol oscillator
- damping and driving in quantum mechanics;
  master equation
- application to quantum vdP oscillator
- spectral function and observed frequency
- two dissipatively coupled vdP oscillators

quantum synchronization

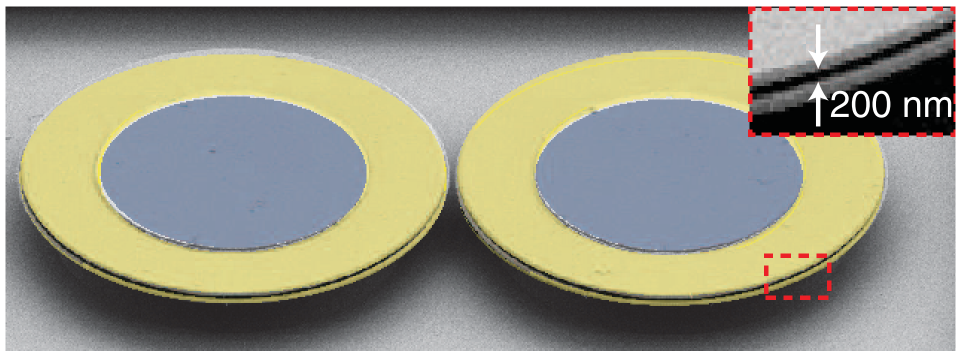
so far only **classical** non-linear systems

synchronization in **quantum** systems:

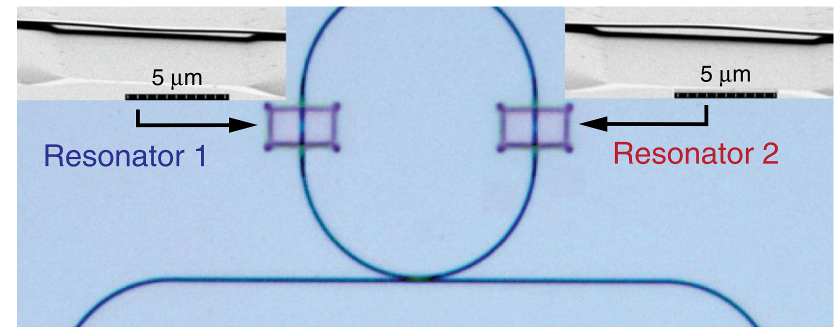
- experimental situation?
- does it exist at all?
- how to quantify and measure it?
- relation to other measures of `quantumness' (entanglement, mutual information,...)

synchronization in microsystems

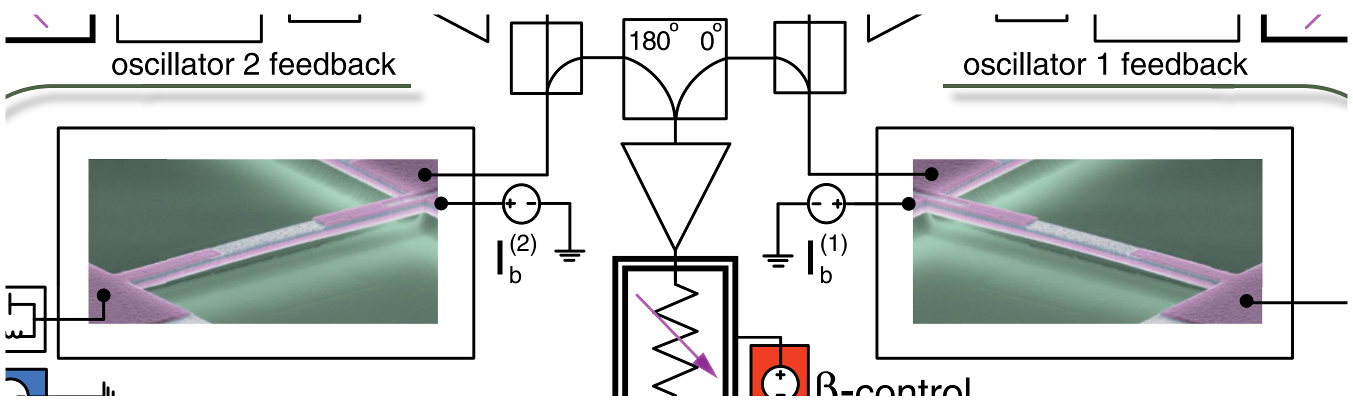
experiments on micro-mechanical systems (not quantum yet)



M. Zhang, G. S. Wiederhecker, S. Manipatruni,
A. Barnard, P. McEuen, and M. Lipson
PRL 109, 233906 (2012); PRL 115, 163902 (2015)



M. Bagheri, M. Poot, L. Fan,
F. Marquardt, and H. X. Tang
PRL 111, 213902 (2013)



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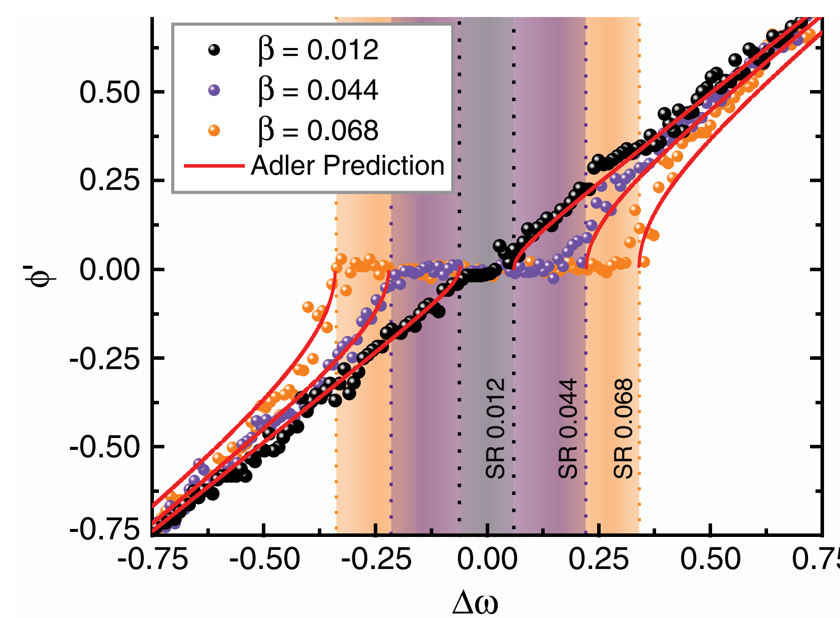
PRL 112, 014101 (2014)



synchronization in microsystems: Roukes et al.

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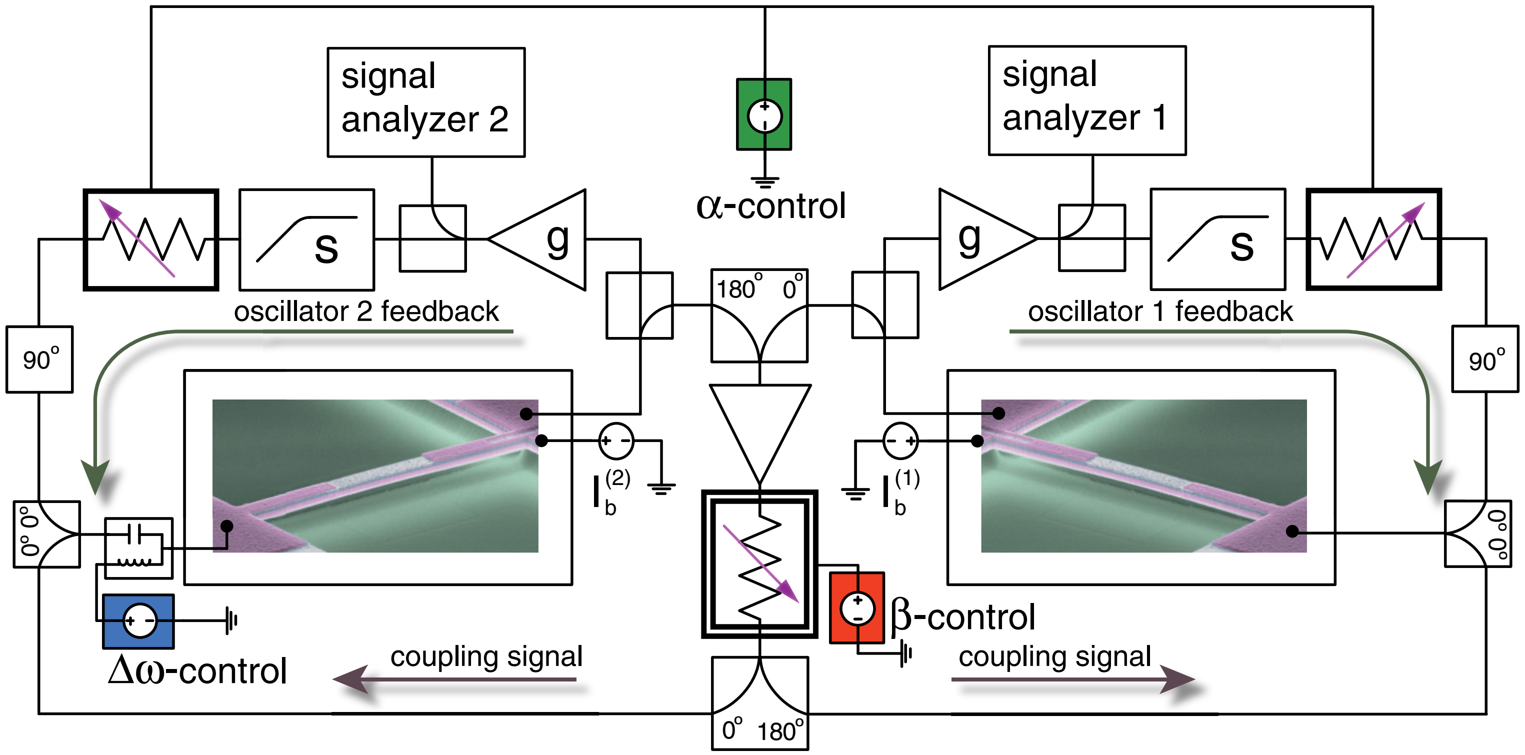
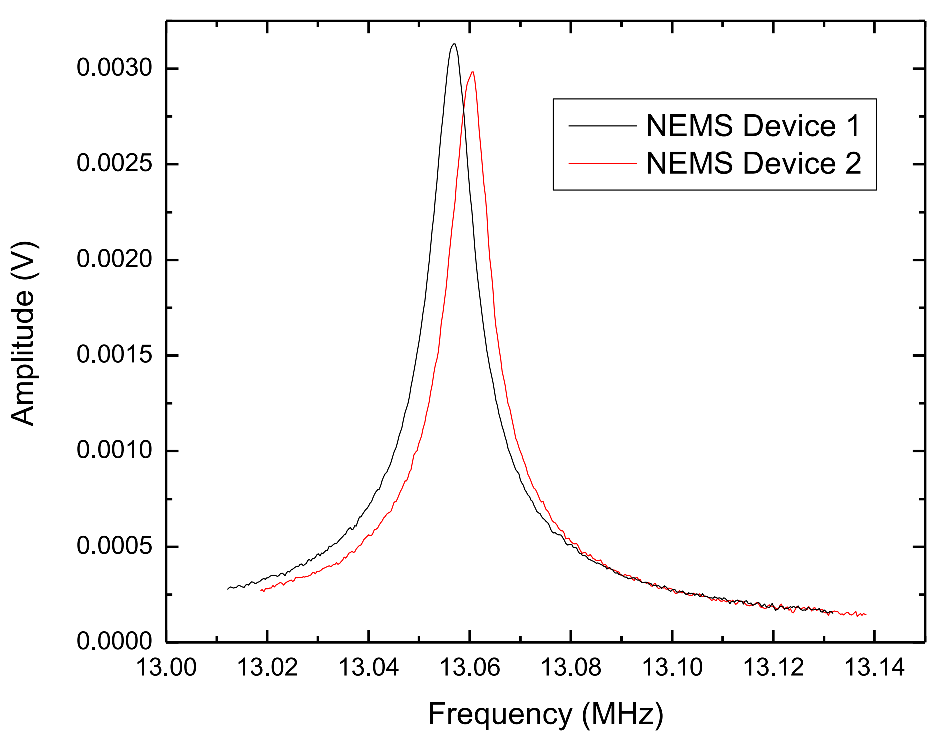


two doubly clamped beams, 10 μm long, 210 nm thick, and 400 nm wide

piezoelectrically actuated, piezo-resistively detected

reactive coupling



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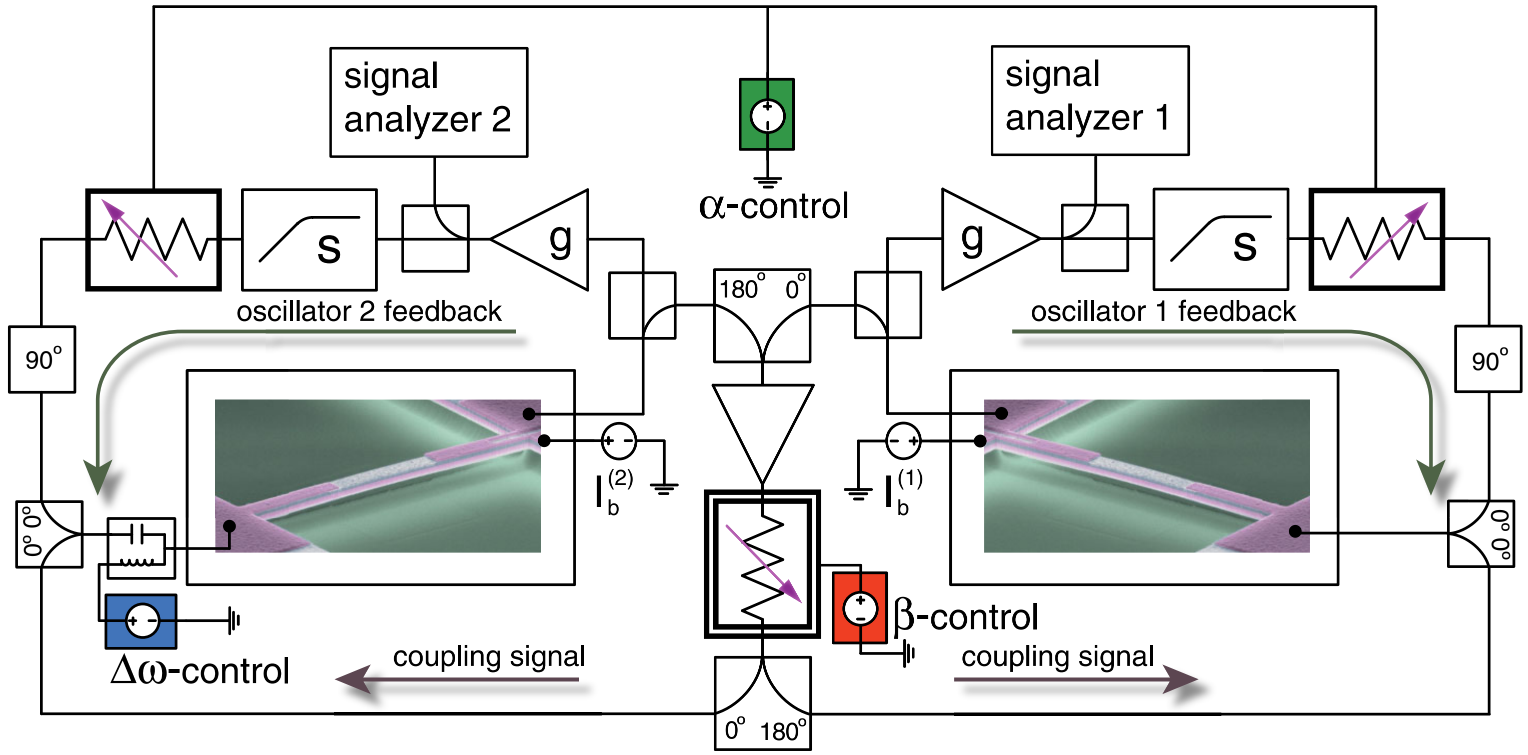

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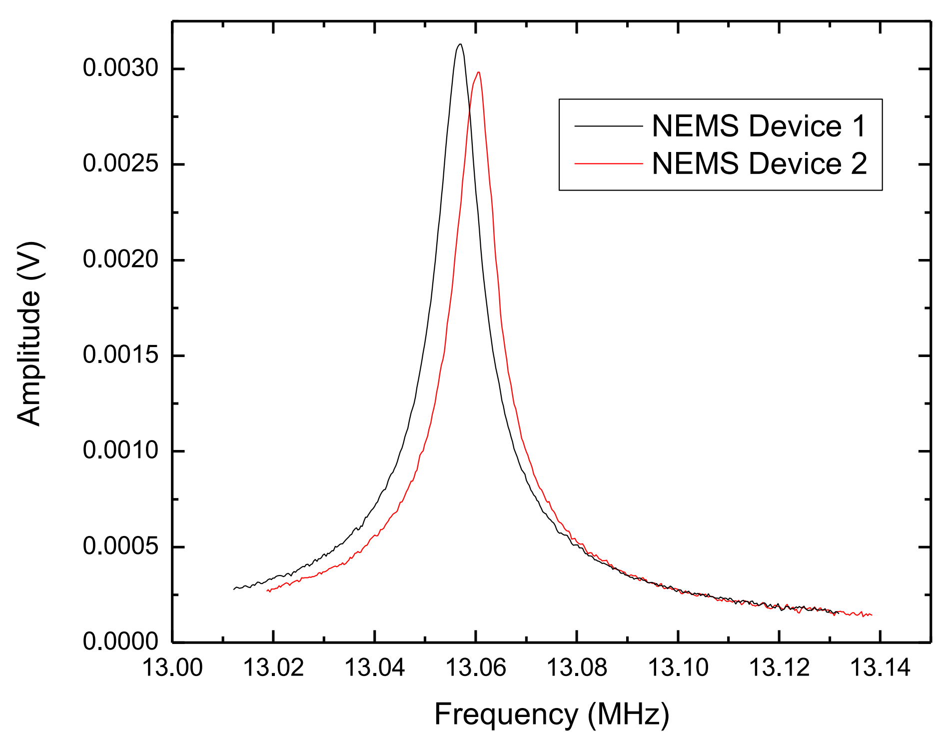
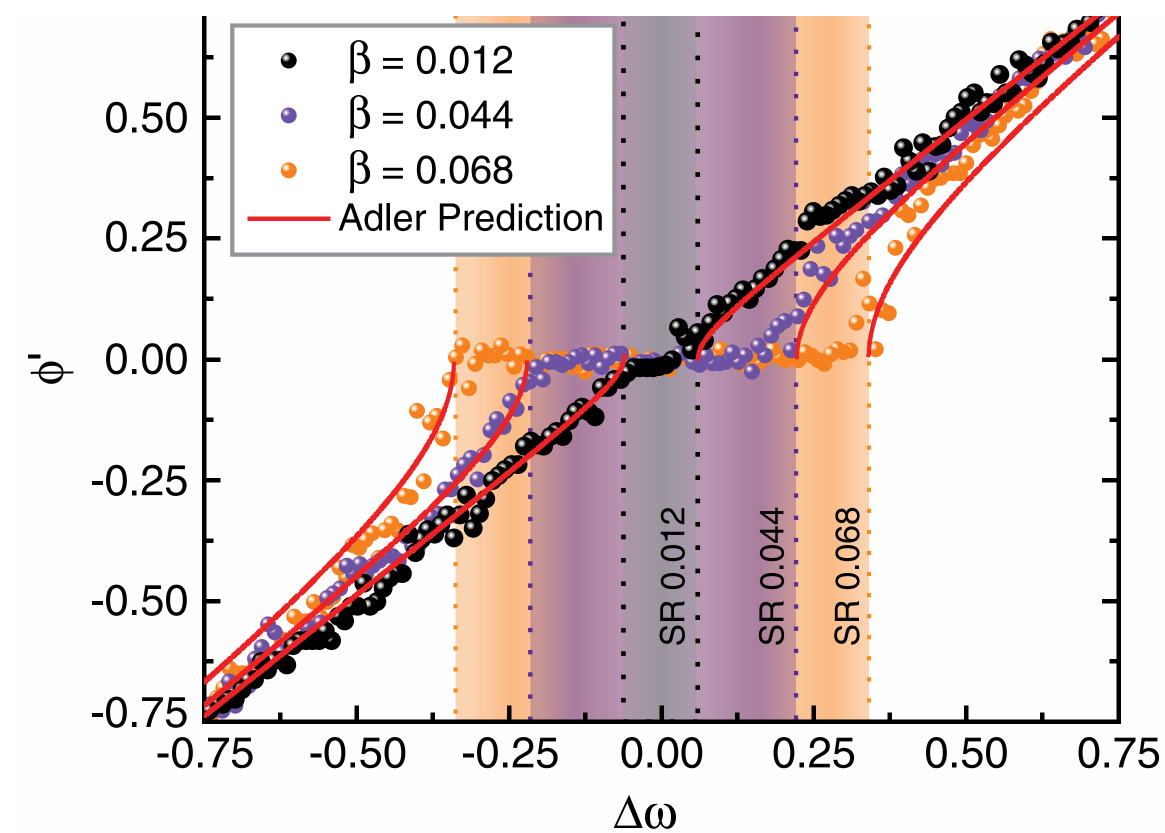




how close to quantum?!

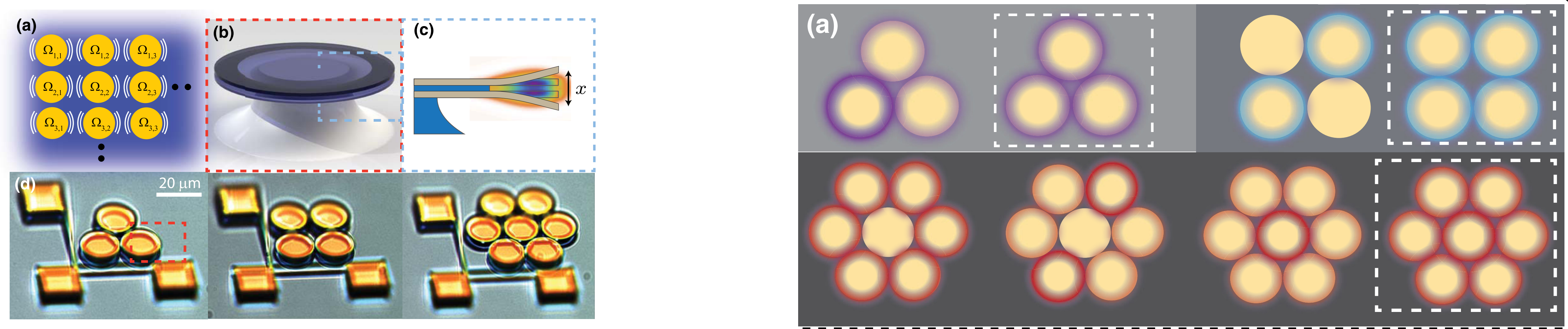
oscillator frequency 13 MHz
ambient temperature

important frequency/temperature/voltage scale

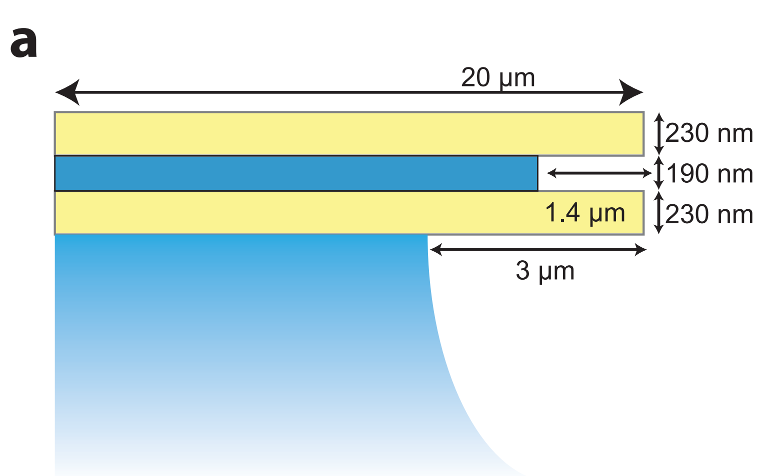
$$20 \text{ GHz} \sim 1 \text{ K} \sim 0.1 \text{ meV}$$

$\Rightarrow$ Roukes experiment is (very) classical

synchronization in microsystems: Lipson et al.



M. Zhang, S. Shah, J. Cardenas, and M. Lipson PRL 115, 163902 (2015)



double-disk optomechanical oscillators composed of two free-standing silicon nitride circular edges

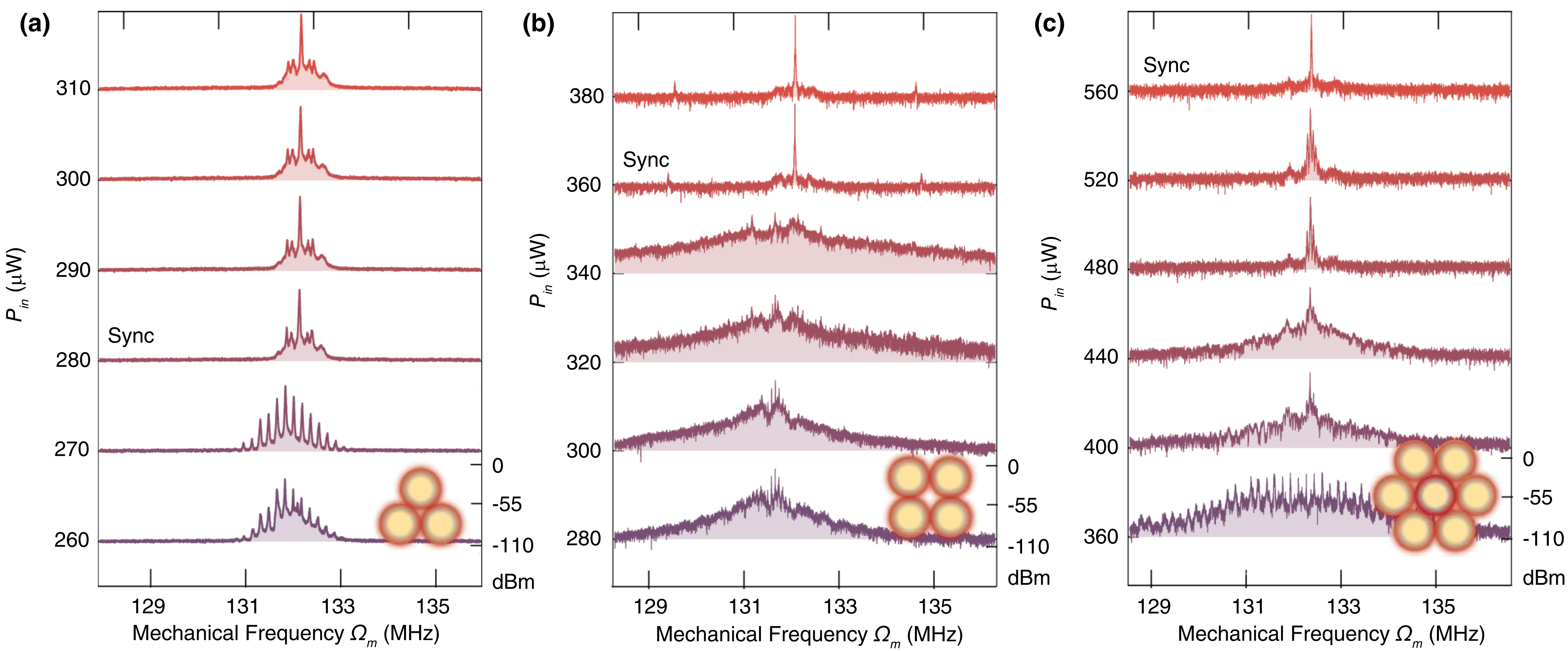
high-Q optical and mechanical modes

mechanical frequency around 132 MHz

coupling by evanescent field via narrow gap (150 nm)

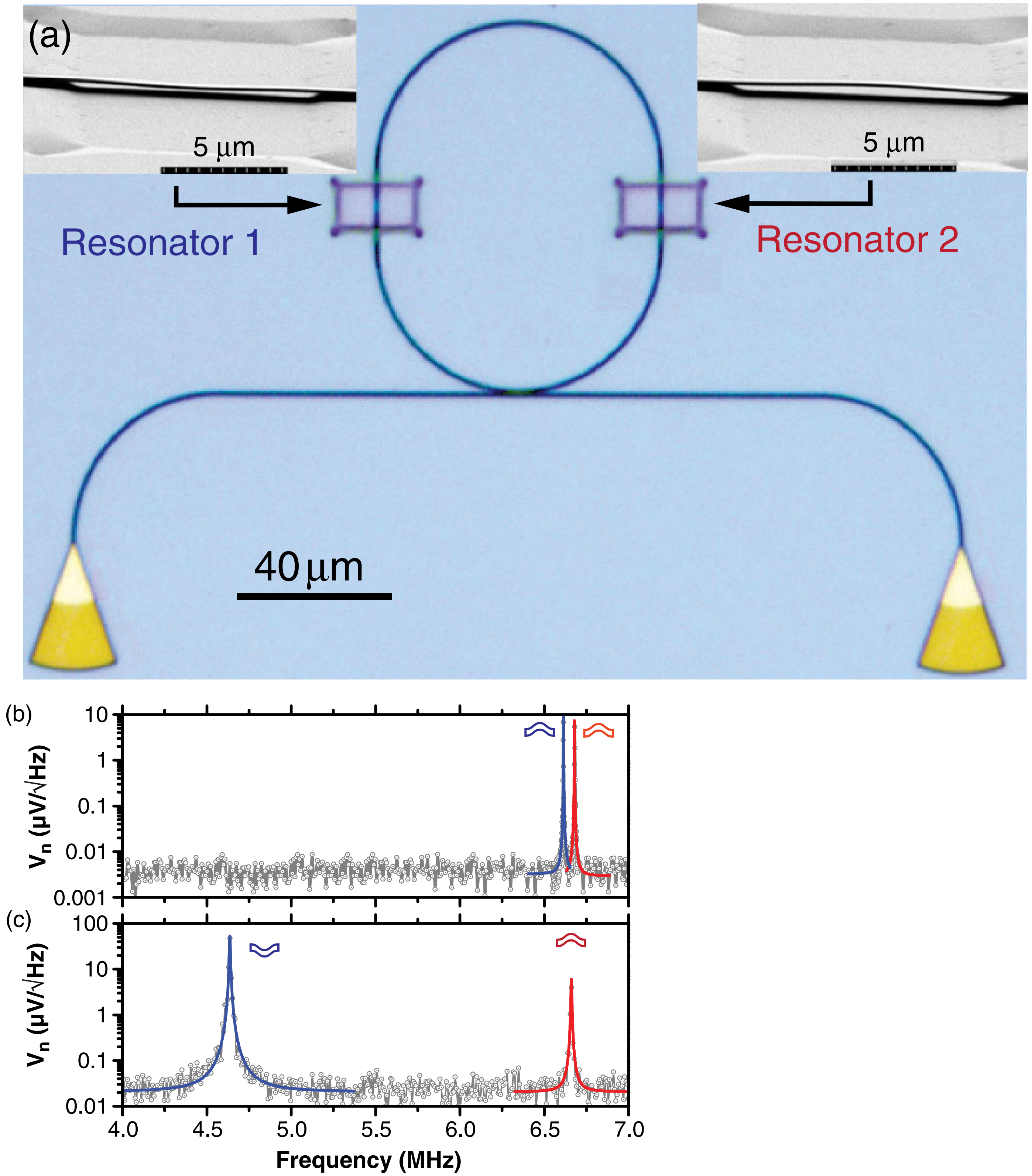
Lipson et al.

optical power spectrum



synchronization in microsystems: Tang et al.

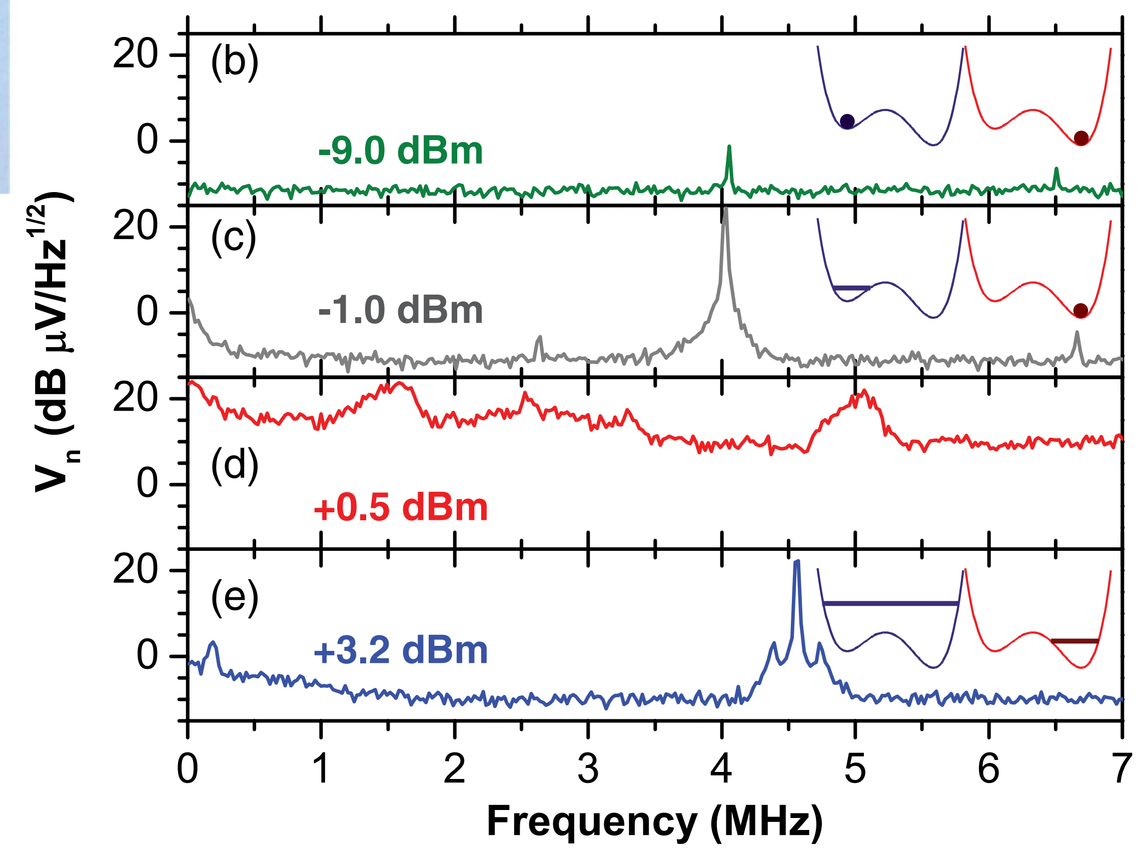
M. Bagheri, M. Poot, L. Fan, F. Marquardt, and H.X. Tang PRL 111, 213902 (2013)



two nanomechanical resonators integrated in an optical racetrack cavity; 10 µm long, 110 nm thick, 500 nm wide

frequency of 6.5 MHz (buckled-up)

V_n: power spectrum of transmitted light

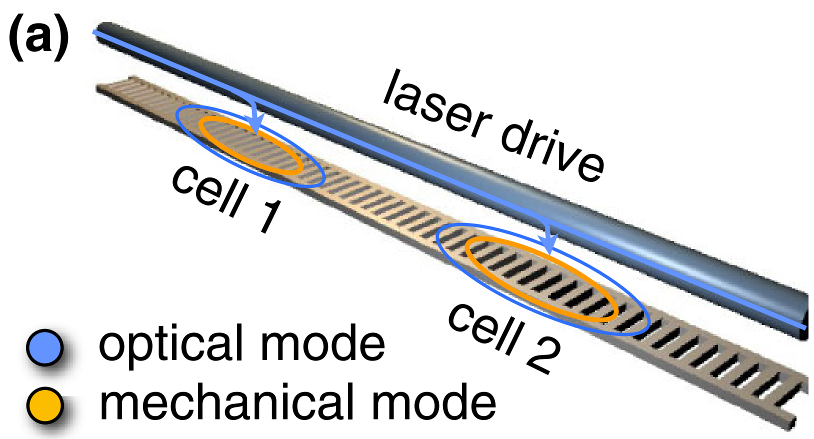


optomechanical systems look favorable

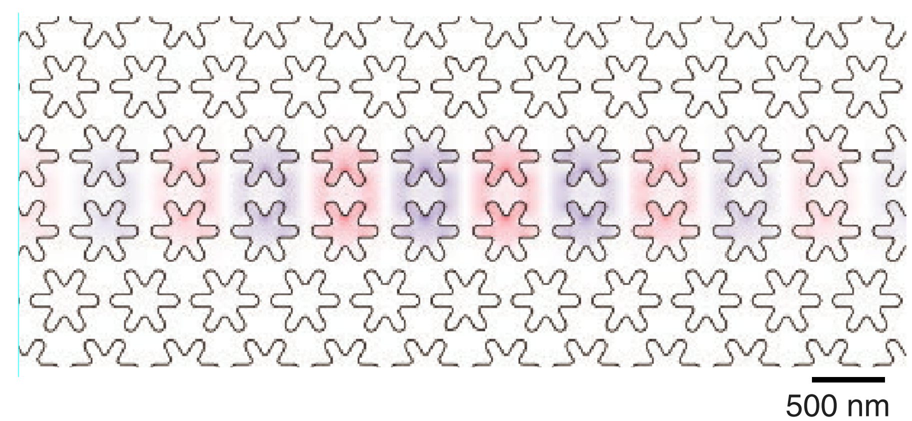
both ground-state cooling and self-sustained oscillations have been demonstrated

J. D.Teufel *et al.*, Nature 475, 359 (2011);
A.H. Safavi-Naeini et al., PRL 108, 033602 (2011);
theory: F. Marquardt et al., PRL 96, 103901 (2006)

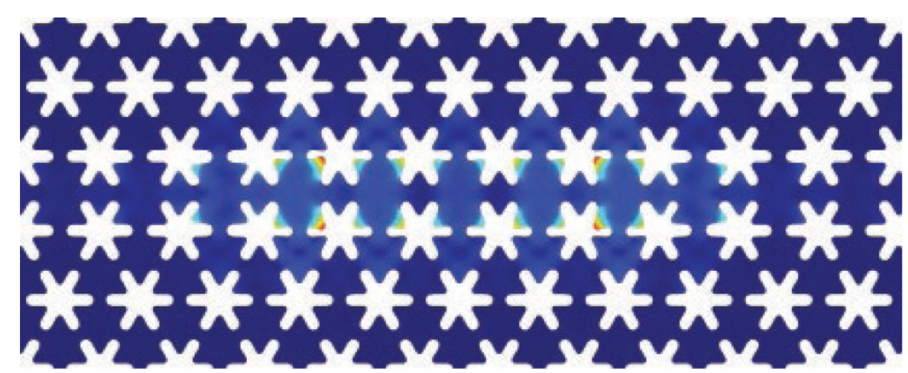
optomechanical arrays
(combined photonic and phononic crystals)



G. Heinrich et al.,
PRL 107, 043603 (2011)



E_y component of electric field

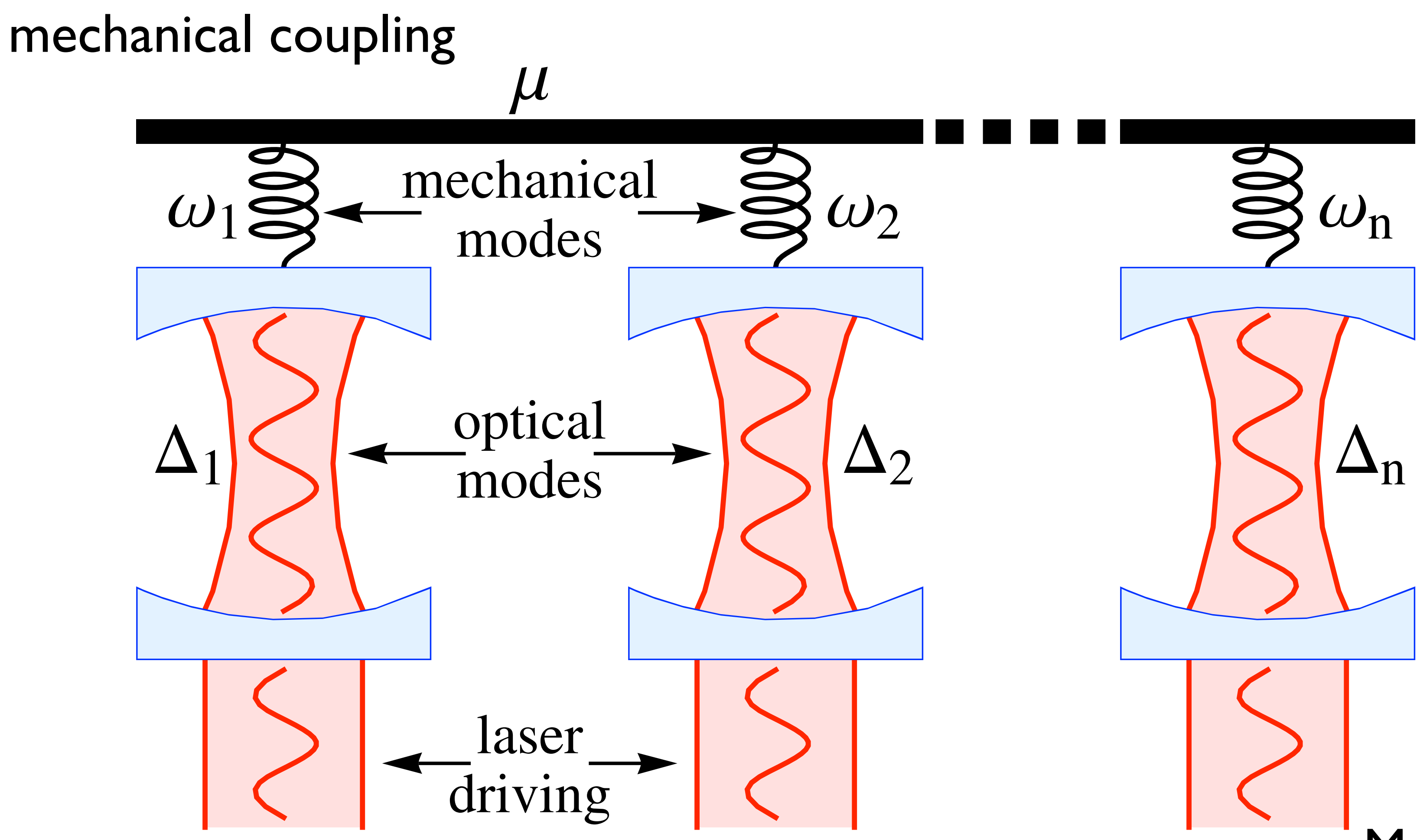


mech. displacement profile

A.H. Safavi-Naeini et al.,
PRL 112, 153603 (2014)

optomechanical systems look favorable

coupled optomechanical cells =
plausible candidate for quantum synchronization experiments



Mari *et al.*, PRL 111, 103605 (2013)

toy model: quantum vdP oscillator

simplest model to study quantum synchronization:
driven quantum van der Pol oscillator

classical case $$\ddot{x} + (-\gamma_1 + \gamma_2 x^2)\dot{x} + \omega_0^2 x = \Omega \cos(\omega_d t)$$

γ_1 negative damping (“battery”)

γ_2 nonlinear damping

Ω strength of the external driving field

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Ω strength of the external driving field

QM formalism

density operator (density matrix) $\rho = \sum_n p_n |n\rangle\langle n|$

p_n probability that state $|n\rangle$ is occupied

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time evolution
(von Neumann equation)

$$i\hbar \frac{d}{dt}\rho(t) = [H, \rho(t)]$$

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how to describe damping?!
how to describe the harmonic drive?

“open quantum system” approach

couple system to an environment
(infinitely many degrees of freedom; continuous spectrum)

H.P. Breuer and F. Petruccione, The theory of open quantum systems, OxfordUP

“open quantum system” approach

couple system to an environment
(infinitely many degrees of freedom; continuous spectrum)

time evolution of ρ_{total} (system + environment) is unitary;
$\rho_{\mathrm{system}} \equiv \rho$ obeys a quantum master equation

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$$\frac{d}{dt}\rho = \frac{-i}{\hbar}[H,\rho] + \sum_j \gamma_j \mathcal{D}_j \rho$$

$\mathcal{D}_j \rho$: dissipator terms

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$\mathcal{D}_j\rho$: dissipator terms

example: damped harmonic oscillator (bath with $T=0$)

$$H = \hbar\omega_0 b^\dagger b; \qquad \mathcal{D}\rho = b\rho b^\dagger - \frac{1}{2}\{b^\dagger b, \rho\}$$

H.P. Breuer and F. Petruccione, The theory of open quantum systems, OxfordUP

example: damped harmonic oscillator

$$H = \hbar\omega_0 b^\dagger b; \qquad \mathcal{D}\rho = b\rho b^\dagger - \frac{1}{2}\{b^\dagger b, \rho\}$$

damping constant γ

$$\frac{d}{dt}\rho = -i\omega_0 b^\dagger b\rho + i\omega\rho b^\dagger b + \gamma b\rho b^\dagger - \frac{1}{2}\gamma(b^\dagger b\rho + \rho b^\dagger b)$$

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matrix elements with Fock states $\quad \rho_{nm} = \langle n|\rho|m\rangle$

$$\frac{d}{dt}\rho_{nm} = (-\tfrac{\gamma}{2} - i\omega_0)n\rho_{nm} + (-\tfrac{\gamma}{2} + i\omega_0)m\rho_{nm} + \gamma\sqrt{n+1}\sqrt{m+1}\rho_{n+1,m+1}$$

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diagonal matrix elements

$$\frac{d}{dt}\rho_{nn} = -\gamma n\rho_{nn} + \gamma(n+1)\rho_{n+1,n+1}$$

exponential decay to stationary solution $\rho_{00} = 1, \quad \rho_{nn} = 0 \quad n \geq 1$

quantum van der Pol equation

Markovian quantum master equation with dissipative terms

$$\frac{d}{dt}\rho(t) = \frac{-i}{\hbar}[H, \rho(t)]$$

$$+ \gamma_1 (b^\dagger \rho b - \frac{1}{2}\{bb^\dagger, \rho\})$$

negative damping,
adds a phonon at rate $\sim \gamma_1$

$$+ \gamma_2 (b^2 \rho b^{\dagger 2} - \frac{1}{2}\{b^{\dagger 2} b^2, \rho\})$$

nonlinear damping,
removes two phonons
at rate $\sim \gamma_2$

harmonic drive

$$H = \hbar\omega_0 b^\dagger b + \frac{\Omega}{x_0} x \cos\omega_d t = \hbar\omega_0 b^\dagger b + \Omega(b^\dagger + b)\cos\omega_d t$$

transformation to the frame of the drive $U = \exp(i\omega_d b^\dagger b t)$

$$H \to U^\dagger H U + i\hbar U^\dagger \dot{U}$$

$$= \hbar(\omega_0 - \omega_d) b^\dagger b + \Omega(b^\dagger + b)$$

driven quantum vdP oscillator

master equation for the density matrix $\Delta = \omega_0 - \omega_d$

drive

non-linear damping

$$\frac{d\rho}{dt} = -i\left[\Delta b^\dagger b + \Omega(b^\dagger + b), \rho\right] + \gamma_1 \mathcal{D}[b^\dagger]\rho + \gamma_2 \mathcal{D}[b^2]\rho$$

detuning

negative damping

$$\mathcal{D}[O]\rho = O\rho O^\dagger - \frac{1}{2}\left\{O^\dagger O, \rho\right\}$$

driven quantum vdP oscillator

master equation for the density matrix $\Delta = \omega_0 - \omega_d$

drive — non-linear damping

$$\frac{d\rho}{dt} = -i\left[\Delta b^\dagger b + \Omega(b^\dagger + b), \rho\right] + \gamma_1 \mathcal{D}[b^\dagger]\rho + \gamma_2 \mathcal{D}[b^2]\rho$$

detuning — negative damping

$$\mathcal{D}[O]\rho = O\rho O^\dagger - \frac{1}{2}\left\{O^\dagger O, \rho\right\}$$

$$\frac{d}{dt}\beta = i\Delta\beta + \frac{\gamma_1}{2}\beta - \gamma_2|\beta|^2\beta - \Omega$$

$$\beta = re^{i\phi} = \langle\hat{b}\rangle$$

classical vdP equation

$$dr/dt = (\gamma_1/2 - \gamma_2 r^2)r - \Omega\cos\phi$$

$$d\phi/dt = \Delta + (\Omega/r)\sin\phi$$

classical and quantum limit

$\gamma_1 \gg \gamma_2$ → negative damping dominates
→ many oscillator levels are populated
→ classical limit

$\gamma_1 \ll \gamma_2$ → nonlinear damping dominates
→ few oscillator levels are populated
(only two for $\gamma_2/\gamma_1 \to \infty$, since
2-phonon processes cannot relax $|1\rangle$)
→ quantum limit

phase-space trajectory and Wigner function

classical phase space trajectory

$$\mathrm{Re}[\beta(t)] \sim x(t) \quad \mathrm{Im}[\beta(t)] \sim p(t)$$

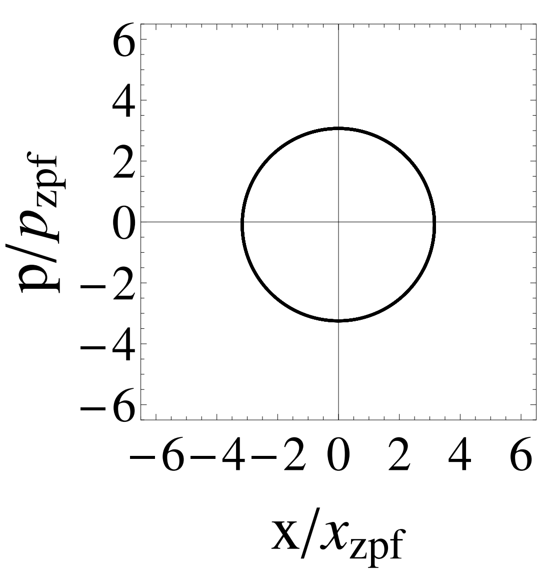
vs.

quantum phase-space

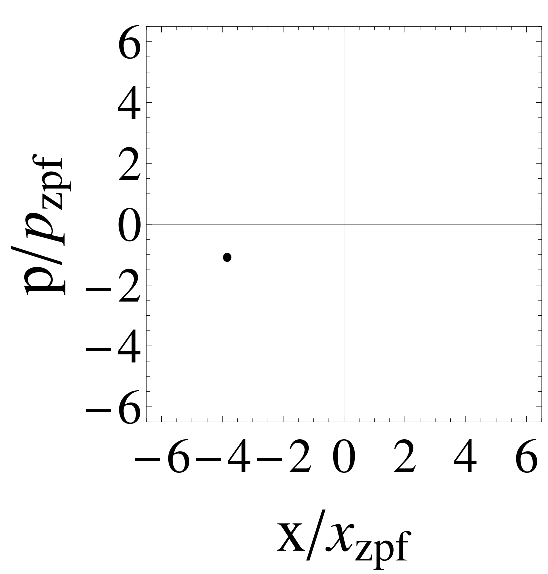
$$W_{ss}(x,p) \sim \int dy e^{-2ipy} \langle x+y|\rho_{ss}|x-y\rangle$$

$\gamma_2 < \gamma_1$

$\Delta/\gamma_1 = 16$

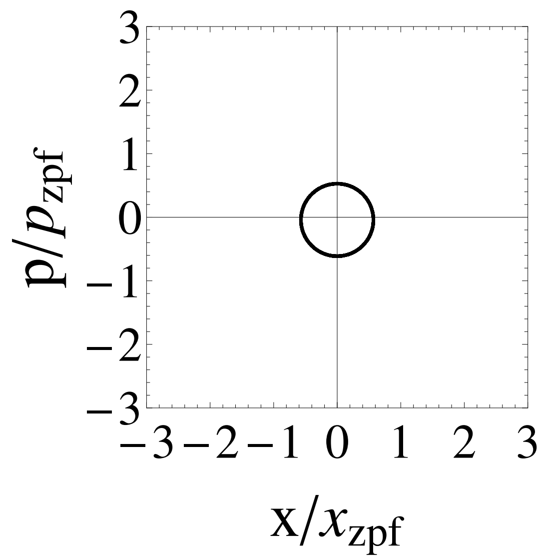


$\Delta/\gamma_1 = 0.1$



$\gamma_2 > \gamma_1$

$\Delta/\gamma_1 = 100$



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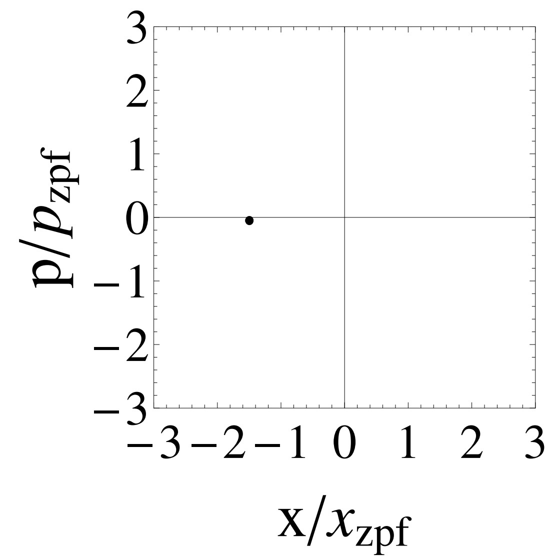

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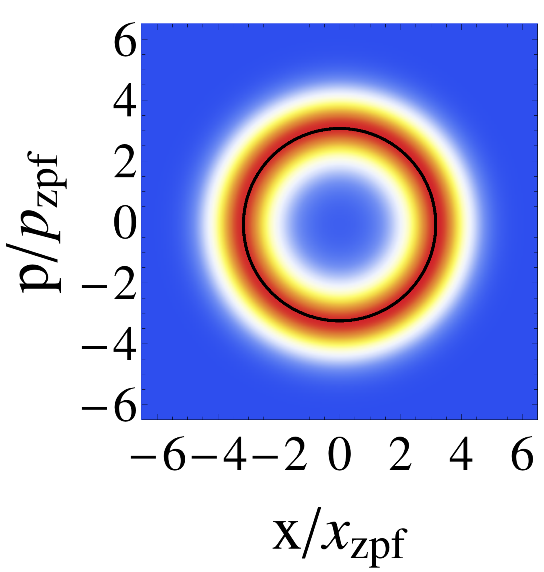

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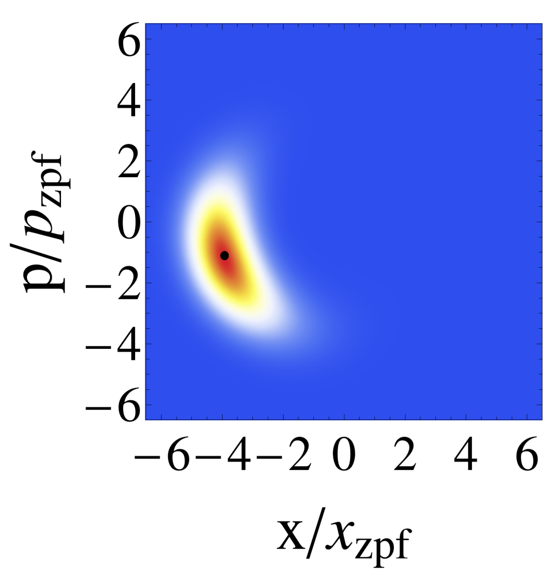

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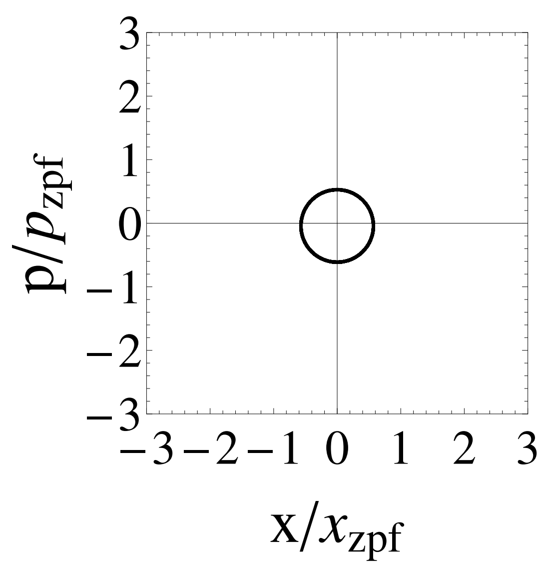

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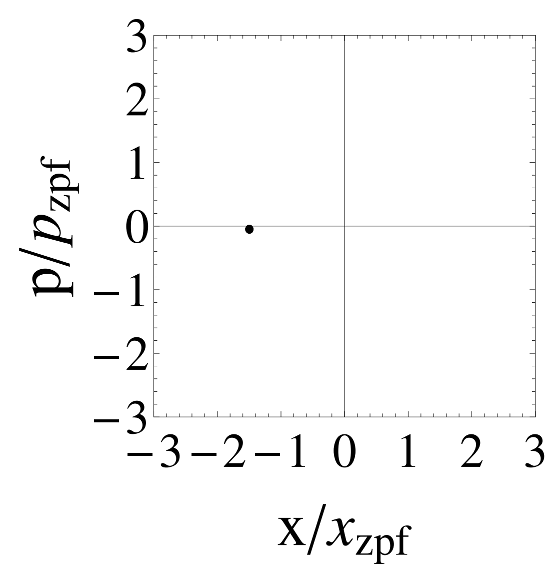

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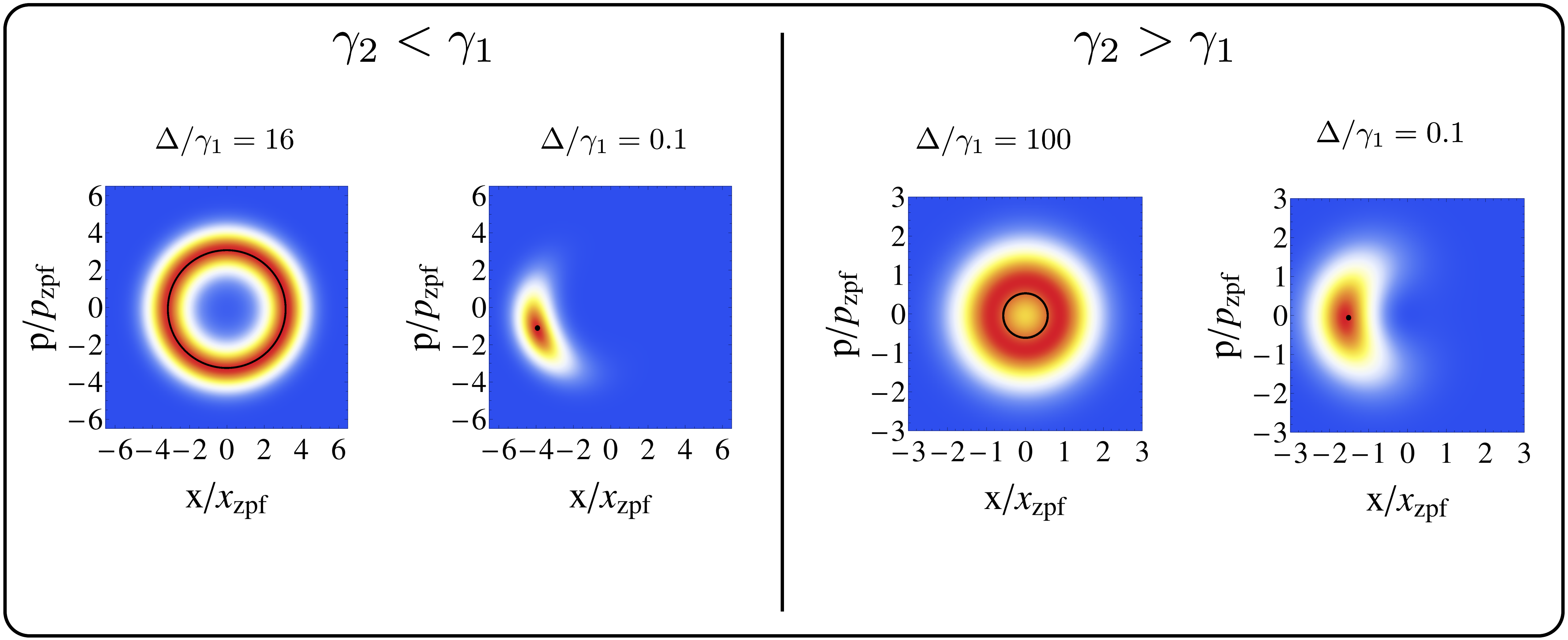

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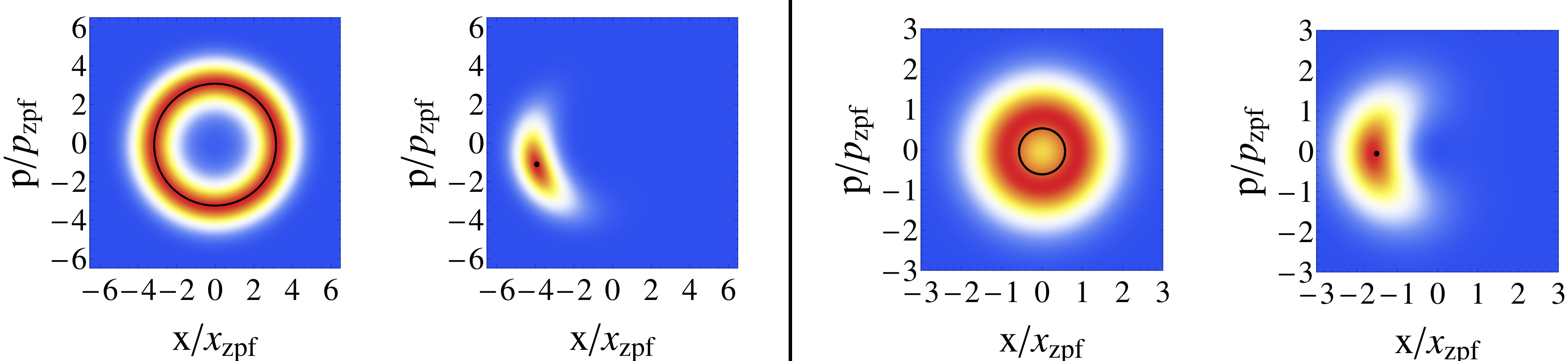

“more classical” $\quad$ “more quantum”

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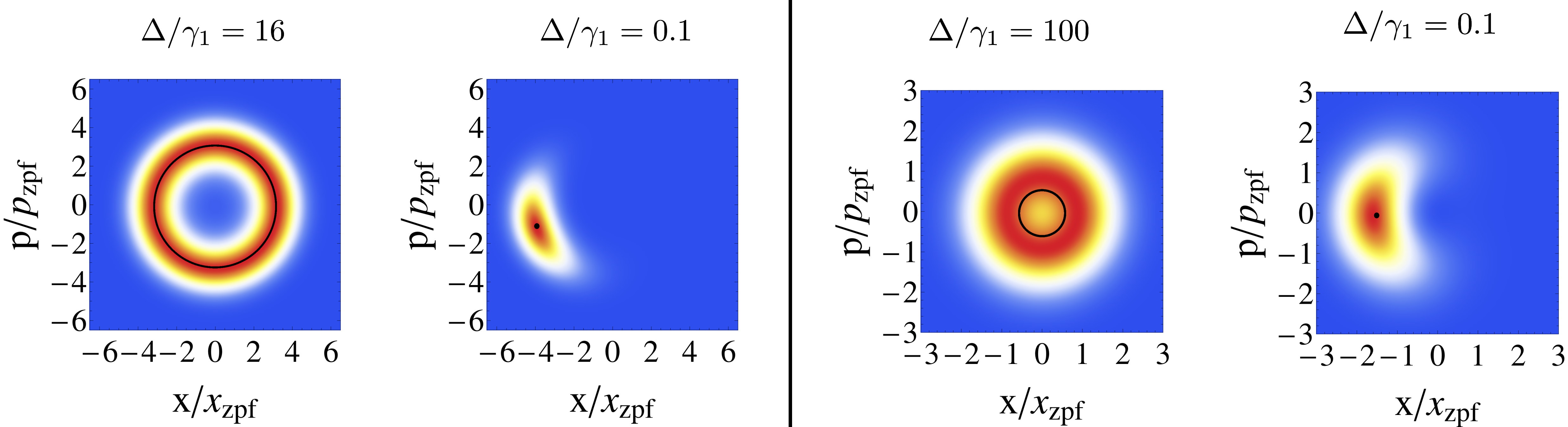

"more classical" | "more quantum"

however:
no information on frequency!

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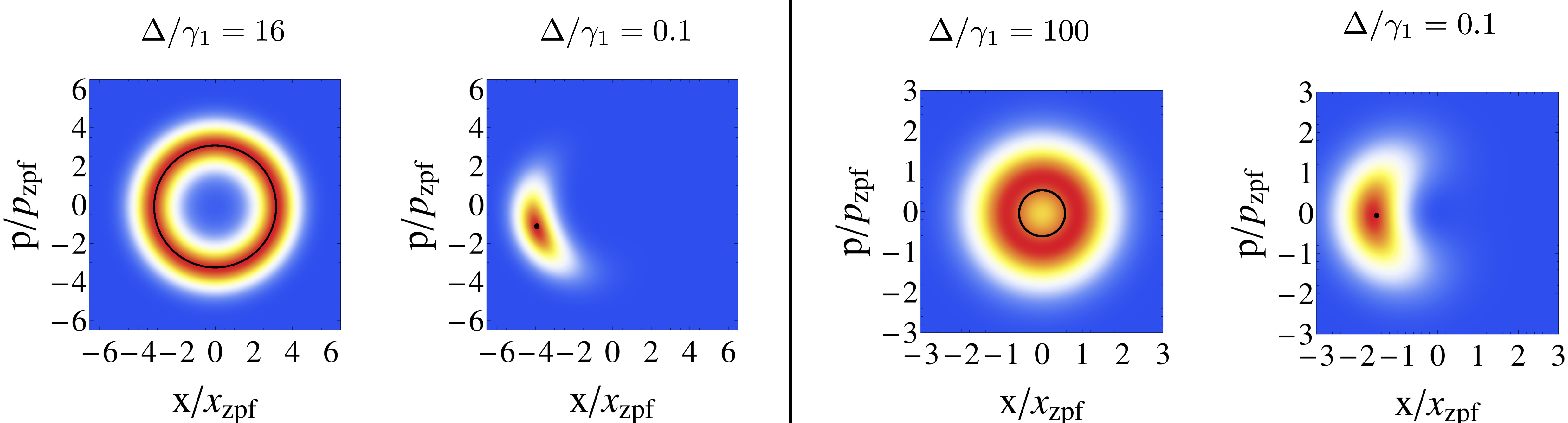

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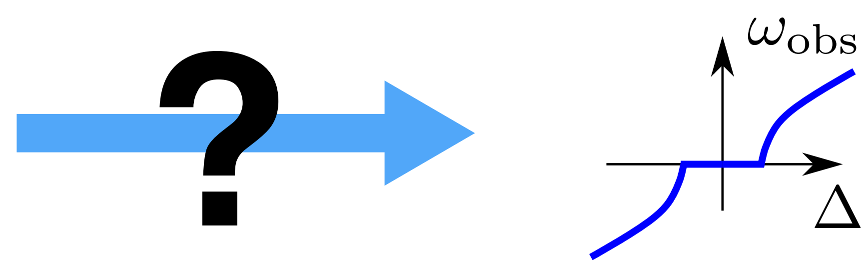

spectral function

spectra and observed frequency:

$$S_{cl}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t} \beta^*(t)\beta(0) \quad \Bigg| \quad S_{qm}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\ \langle \hat{b}^\dagger(t)\hat{b}(0)\rangle$$

see also Cresser et al., PRA 1982; Schleich and Scully PRA 1988

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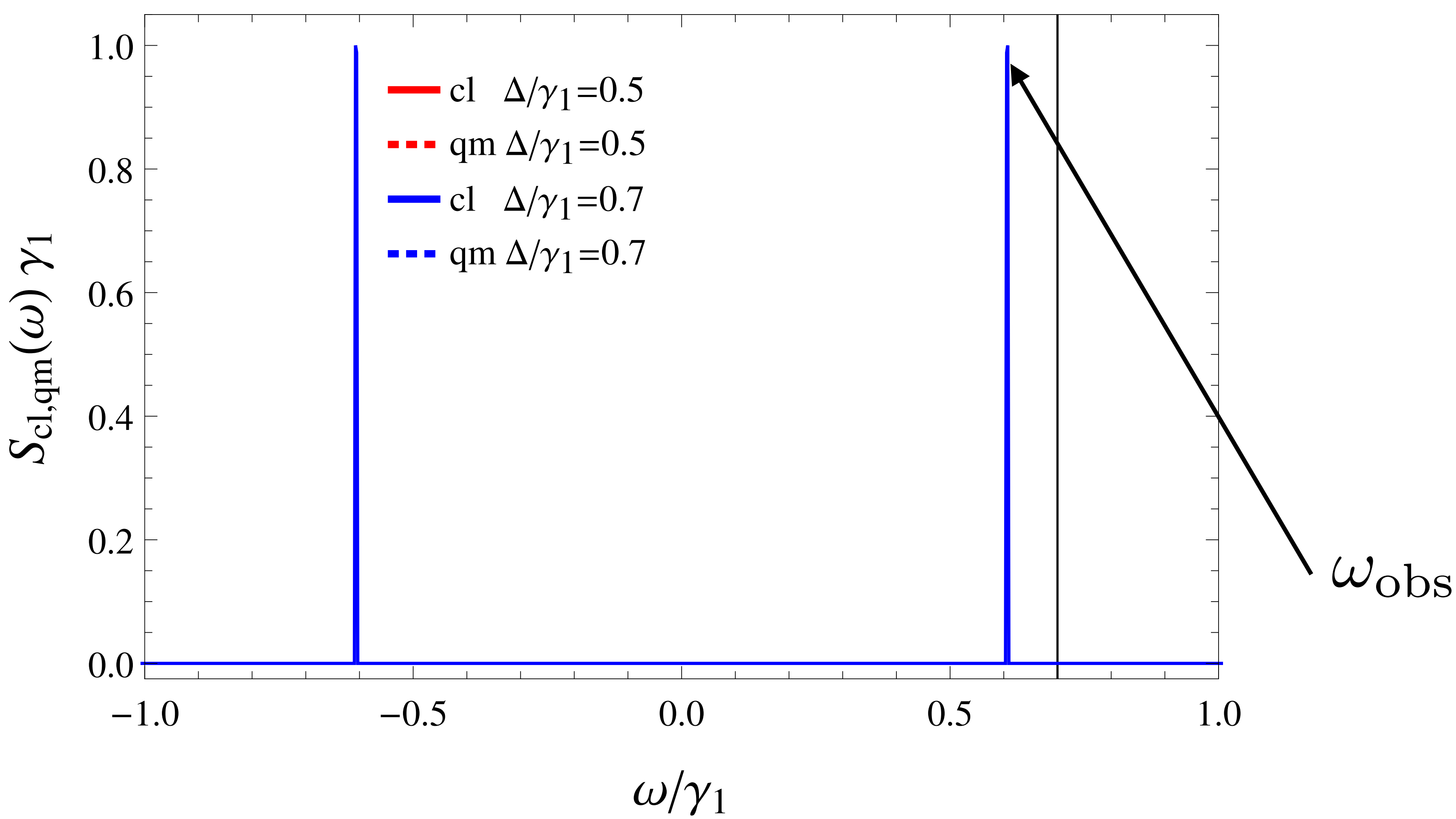
$\Omega/\gamma_1 = 1$

$\gamma_2/\gamma_1 = 0.1$



spectral function

spectra and observed frequency:

$$S_{cl}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t} \beta^*(t)\beta(0) \quad \Big| \quad S_{qm}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\ \langle \hat{b}^\dagger(t)\hat{b}(0)\rangle$$

see also Cresser et al., PRA 1982; Schleich and Scully PRA 1988

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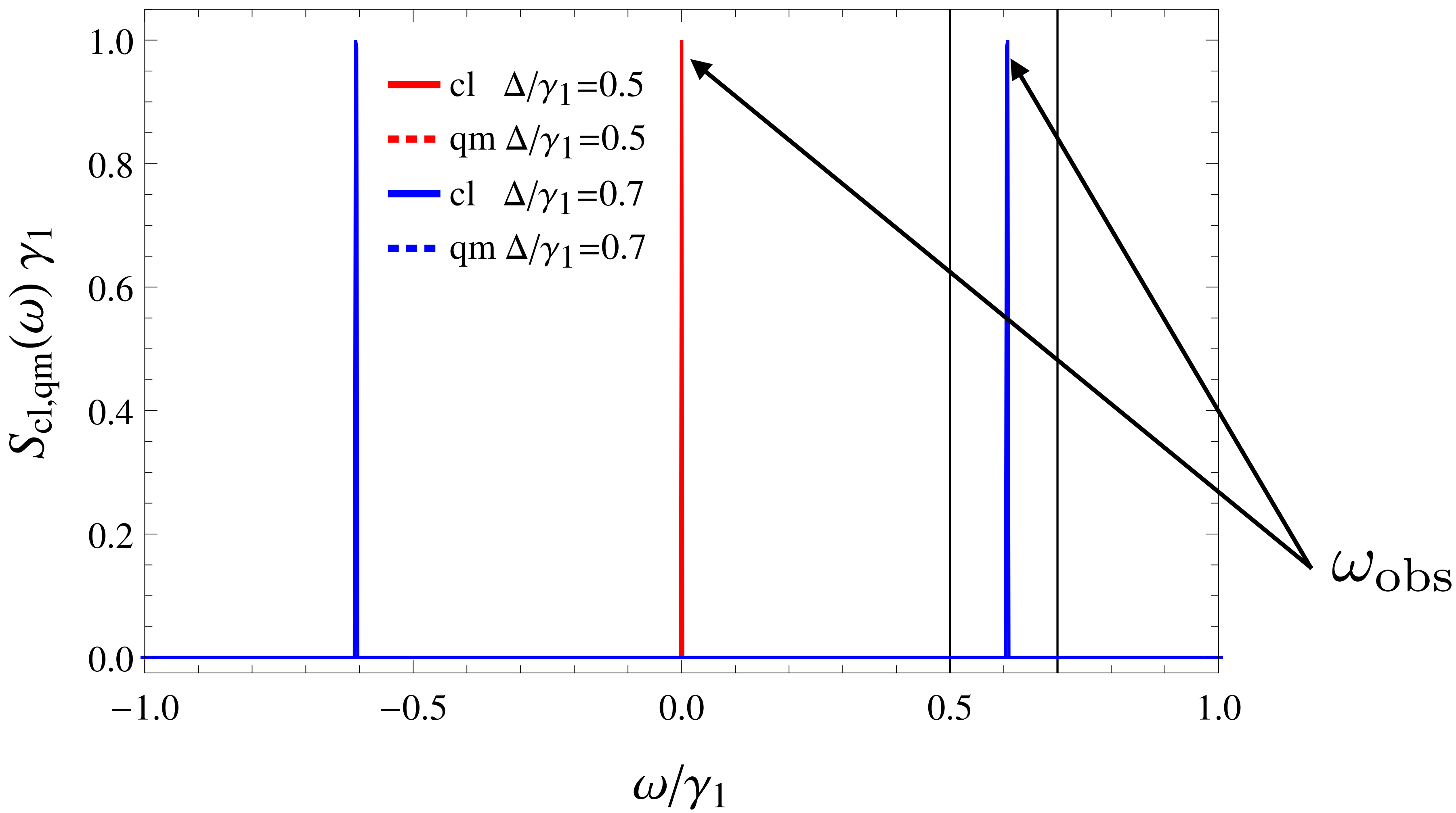

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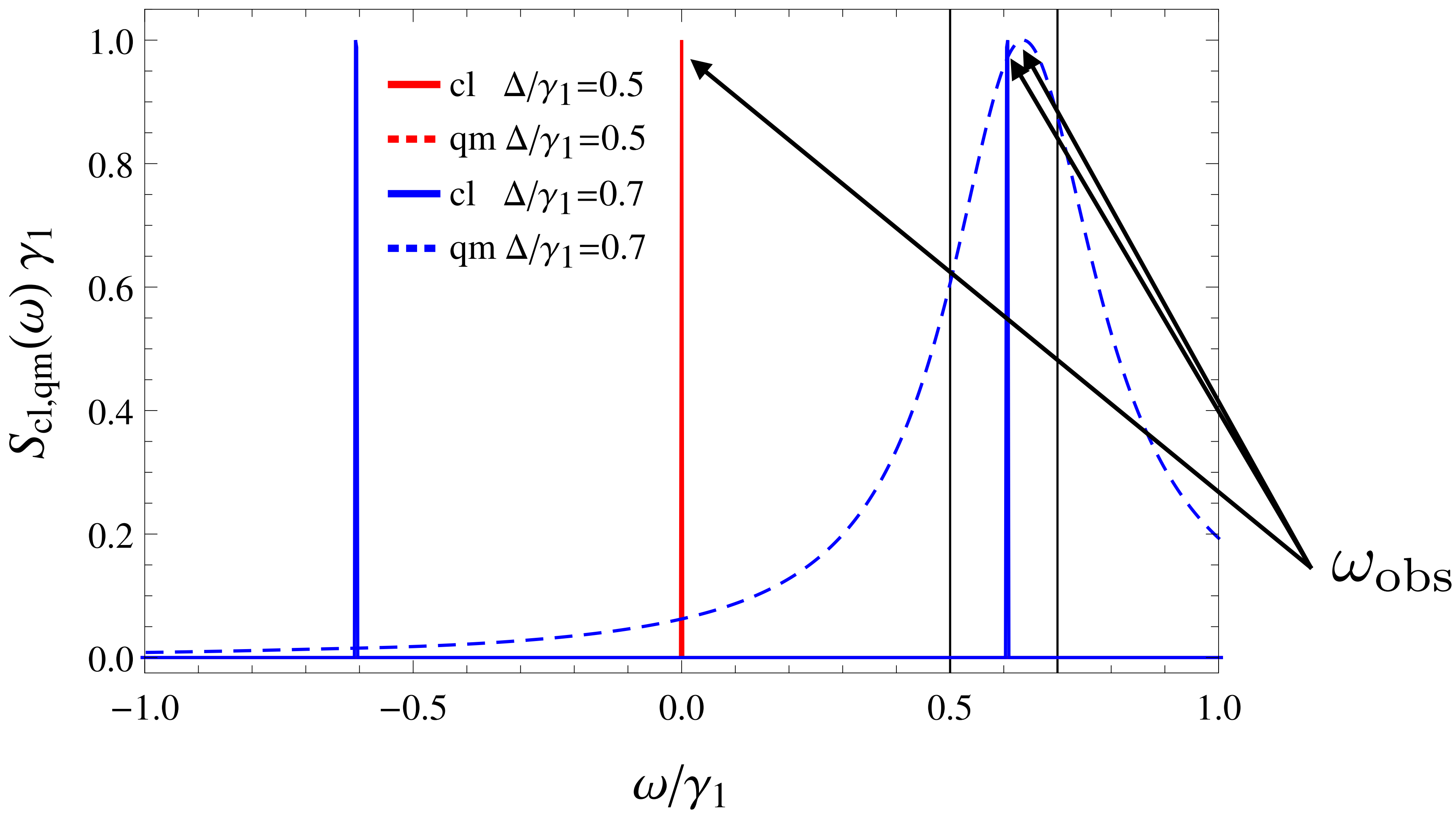

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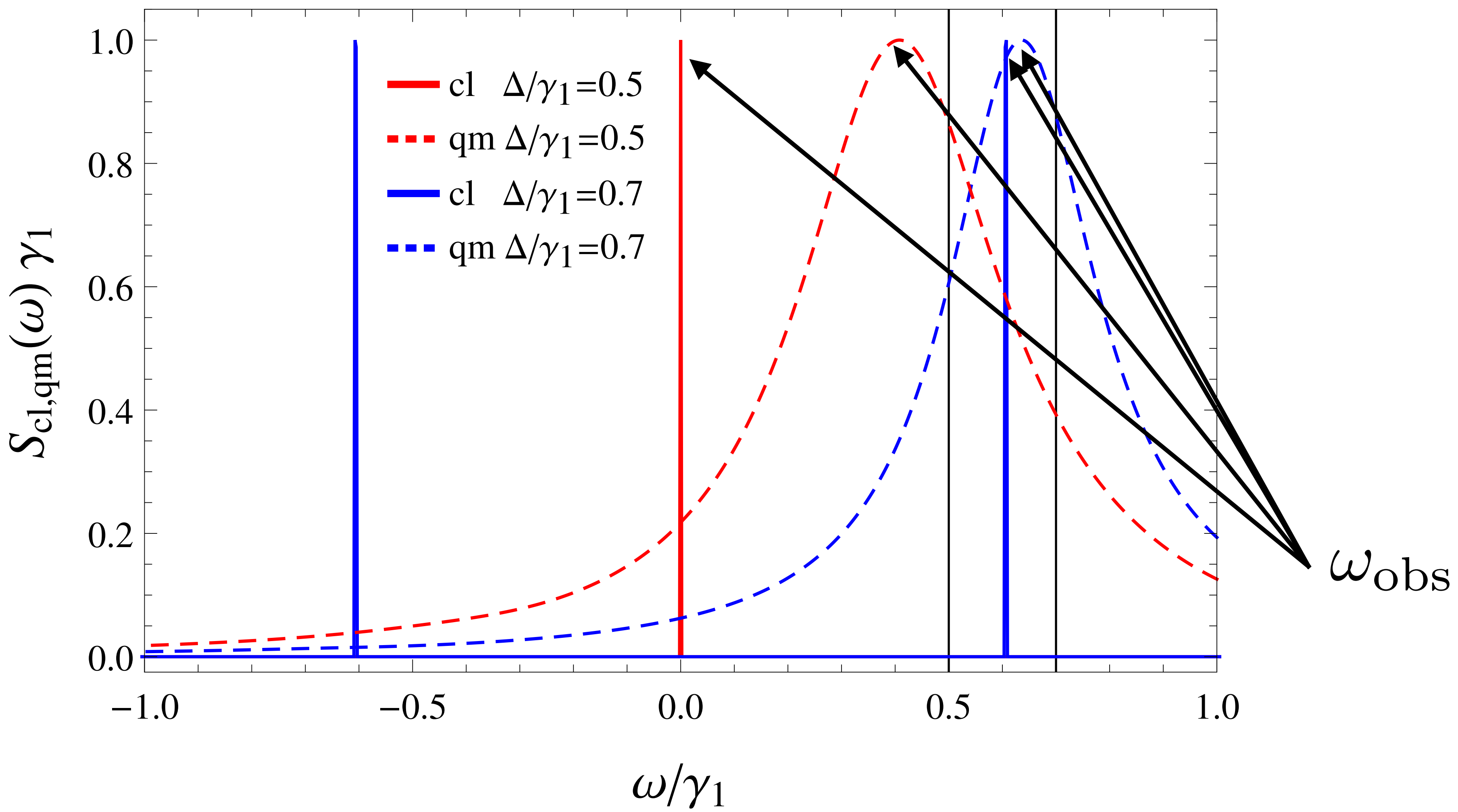
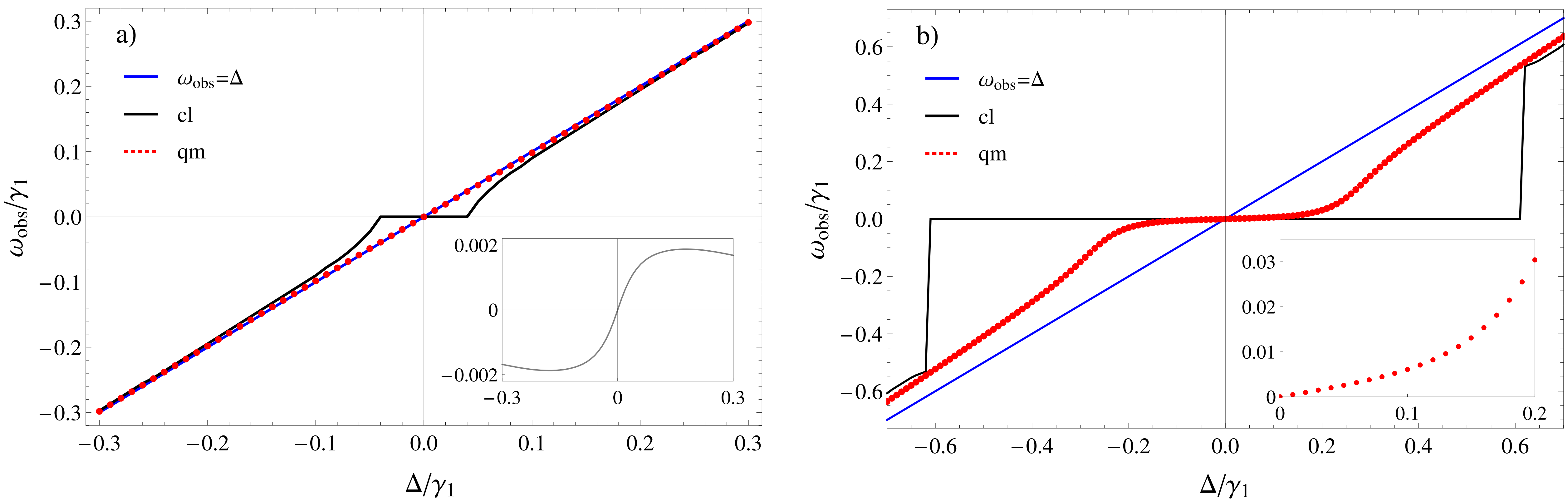


observed frequency vs. detuning

$$\gamma_2/\gamma_1 = 0.1$$

$\Omega/\gamma_1 = 0.1$ $\Omega/\gamma_1 = 1$



weak entrainment strong entrainment

no plateau → absence of exact frequency locking

interpretation: quantum noise suppresses synchronization

S. Walter, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. 112, 094102 (2014)

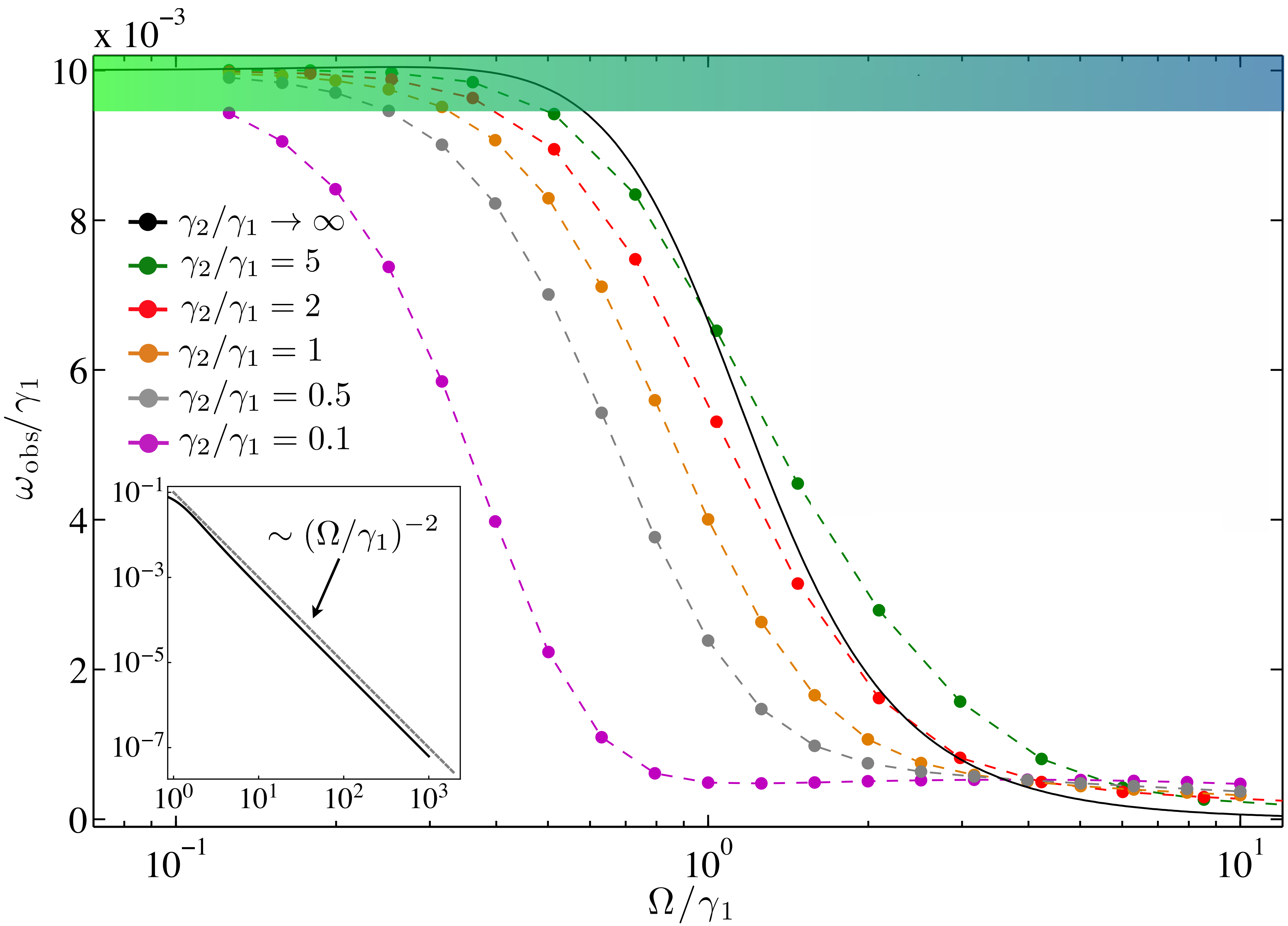
observed frequency vs. driving strength

$\Delta/\gamma_1 = 0.01$

weak entrainment

crossover

strong entrainment



black line: analytical result

physical realization

engineer dissipative processes in a "membrane-in-the-middle" optomechanical setup using two lasers

J.D. Thompson, B.M. Zwickl, A.M. Jayich, F. Marquardt, S.M. Girvin, and J.G.E. Harris, Nature 452, 72 (2008)

quadratic optomechanical coupling to favor two-phonon processes

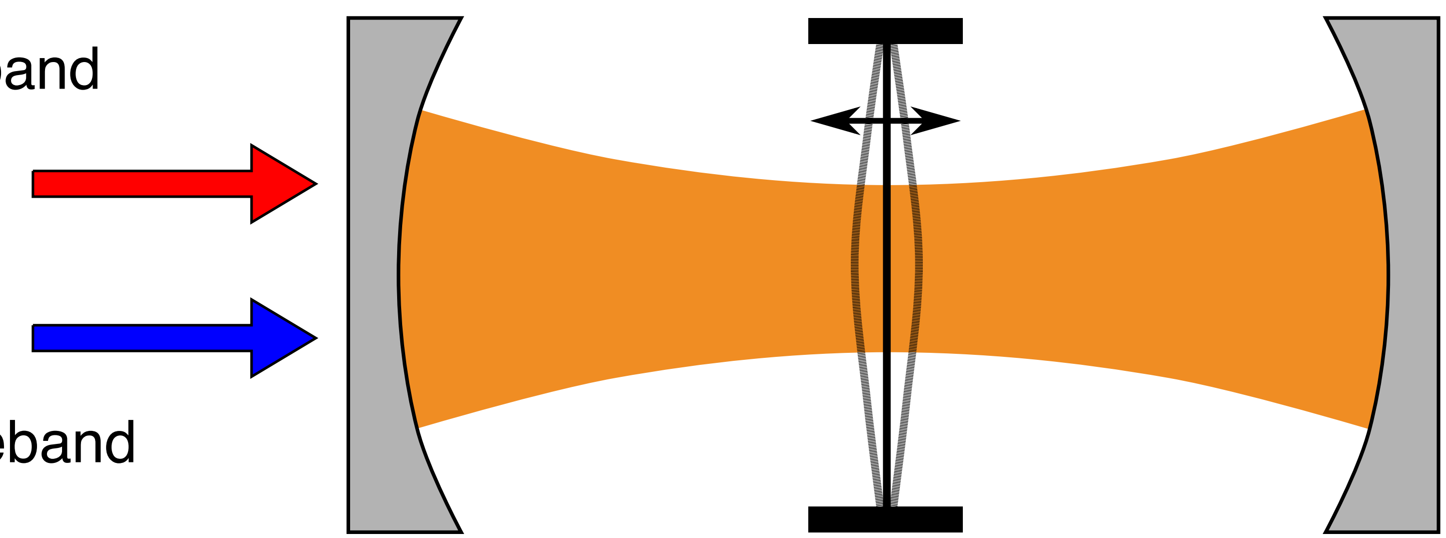
A. Nunnenkamp, K. Børkje, J.G.E. Harris, and S.M. Girvin, PRA 82, 021806(R) (2010)

detuned to the **red** two-phonon sideband

➡ non-linear damping $\gamma_2 \mathcal{D}[\hat{b}^2]$

detuned to the **blue** one-phonon sideband

➡ negative damping $\gamma_1 \mathcal{D}[\hat{b}^\dagger]$



two coupled oscillators

different types of coupling:

two coupled oscillators

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reactive = via a term in the Hamiltonian, e.g.

$\frac{g}{2}(x_1 - x_2)^2$ classical

$g(b_1^\dagger b_2 + b_2^\dagger b_1)$ quantum

two coupled oscillators

different types of coupling:

reactive = via a term in the Hamiltonian, e.g.

$$\frac{g}{2}(x_1 - x_2)^2$$ classical

$$g(b_1^\dagger b_2 + b_2^\dagger b_1)$$ quantum

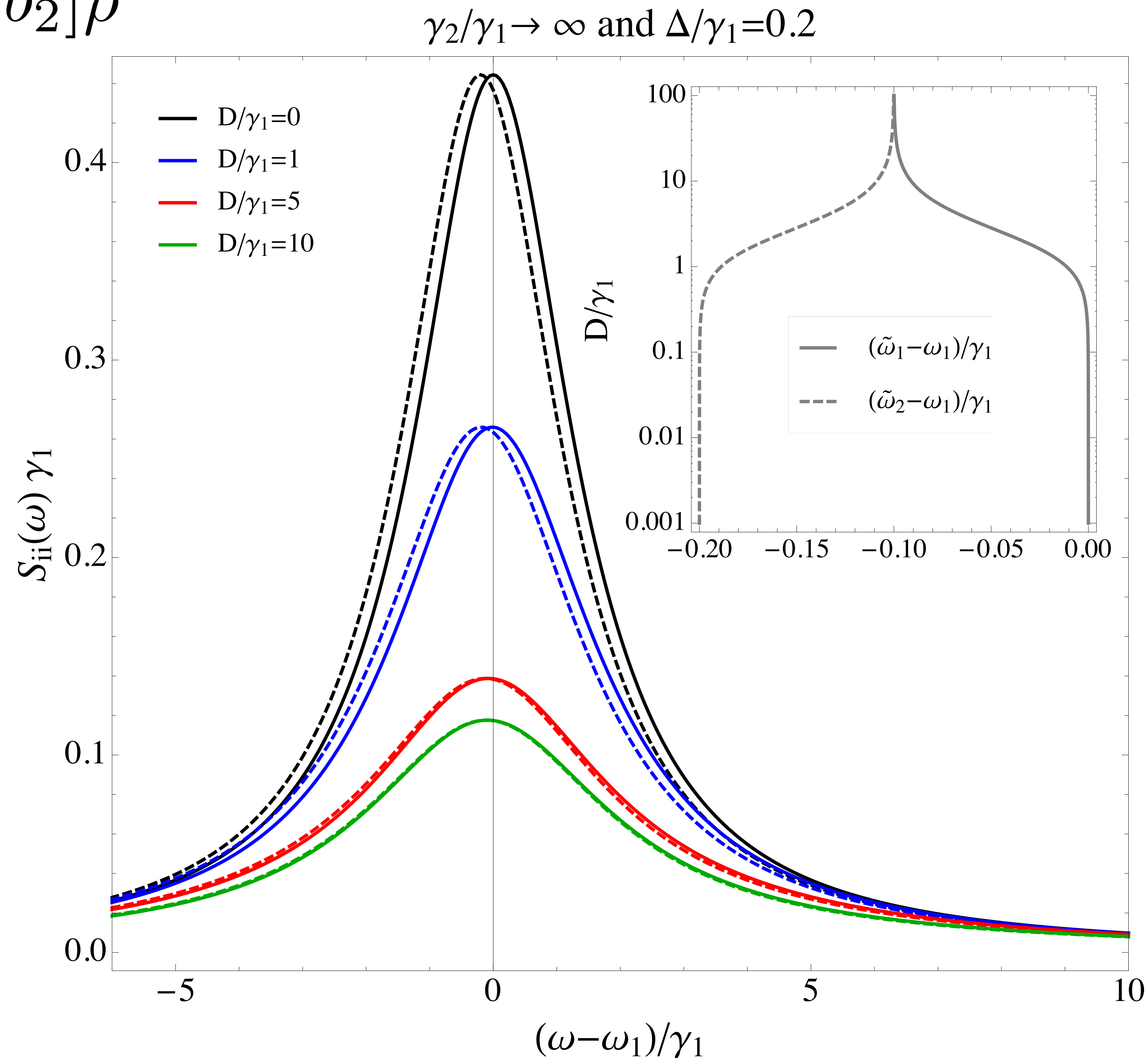
dissipative = via a term in the equation of motion, e.g.

$$D(\dot{x}_2 - \dot{x}_1)$$ classical

$$D\mathcal{D}[b_1 - b_2]\rho$$ quantum

two dissipatively coupled oscillators

$$\dot{\rho} = \mathcal{L}_0\rho + D\mathcal{D}[b_1 - b_2]\rho$$



S. Walter, A. Nunnenkamp, and C. Bruder, Ann. Phys. (Berlin) 527, 131(2015)

conclusion

- experiments in microsystems are approaching the quantum threshold
- toy model: driven quantum van der Pol oscillator
- phase space plots: hint towards quantum synchronization
- power spectrum as important observable
- absence of true frequency locking due to quantum noise
- similar for two dissipatively coupled vdP oscillators

appendix

collaborators

Ehud Amitai

Niels Lörch

Andreas Nunnenkamp, Basel → Cambridge

Stefan Walter, Basel → Erlangen

S. Walter, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. 112, 094102 (2014)
S. Walter, A. Nunnenkamp, and C. Bruder, Ann. Phys. (Berlin) 527, 131 (2015)
N. Lörch, E. Amitai, A. Nunnenkamp, and C. Bruder, arXiv:1603.01409

two dissipatively coupled oscillators

possible realization:
couple two vdP oscillators to a common optical mode c

$$\chi = \rho_c \otimes \rho$$

$$\dot{\chi} = -i\,[H_{tot}, \chi] + \kappa \mathcal{D}[c]\chi + \sum_{i=1,2} \gamma_1^{(i)} \mathcal{D}[b_i^\dagger]\chi + \gamma_2^{(i)} \mathcal{D}[b_i^2]\chi$$

$$H_{tot} = -\Delta_c c^\dagger c \; + \sum_{i=1,2} \omega_i\, b_i^\dagger b_i + G_i (c + c^\dagger)(b_i + b_i^\dagger)$$

S. Walter, A. Nunnenkamp, and C. Bruder, Ann. Phys. (Berlin) 527, 131(2015)

two dissipatively coupled oscillators

possible realization:
couple two vdP oscillators to a common optical mode c

a) $\omega_i \gg \kappa \gg G_i, \gamma_1^{(i)}, \gamma_2^{(i)}$ eliminate mode c

b) choose $\Delta_c \approx -\omega_{1,2}$

c) choose $G_1 = G_2 = G$

$$\dot{\rho} = \sum_{i=1,2} -i\left[\omega_i b_i^\dagger b_i, \rho\right] + \gamma_1^{(i)} \mathcal{D}[b_i^\dagger]\rho + \gamma_2^{(i)} \mathcal{D}[b_i^2]\rho + \frac{4G^2}{\kappa}\mathcal{D}[b_1 + b_2]\rho$$

S. Walter, A. Nunnenkamp, and C. Bruder, Ann. Phys. (Berlin) 527, 131(2015)