III. From Majorana to Parafermions

Daniel Loss
University of Basel
Switzerland

In collaboration with Jelena Klinovaja, Basel

$$: \text{Swiss NSF, Nano Basel, Quantum ETH/Basel, EU}$$
Outline

• Topological quantum computing

• Majorana fermions in ‘super-semi’ nanowires and chains

• Majorana and spin qubits ➔ 2D surface code

• Next generation: Parafermions in double nanowires ➔ can get nearly universal TQC including CNOT
At $T=0$: TQC protected against all errors by gap

Motivation: Topological Quantum Computing

Kitaev 2003

braiding of quasiparticles with non-Abelian statistics

Majoranas, parafermions, Fibonacci fermions, …
Motivation: Topological Quantum Computing

Kitaev 2003

At T=0: TQC protected against all errors by gap
At T>0: errors occur with finite probability
→ need continuous quantum error correction for TQC

Wootton, Burri, Iblisdir, and Loss, PRX 4, 011051 (2014)
Brell, Burton, Dauphinais, Flammia, and Poulin, PRX 4, 031058 (2014)
Pedrocchi, Bonesteel, and DiVincenzo, PRL 115, 120402 (2015); PRB 92, 115441 (2015)
Hutter and Wootton, PRA 93, 042327 (2016)
Brown, DL, Pachos, Self, and Wootton, Rev. Mod. Phys. 88, 045005 (2016)
decay of ground state
\[ b_0^\dagger |\Omega\rangle \rightarrow |k\rangle = b_k^\dagger |\Omega\rangle \]
due to local gate fluctuations \( \mu_{\text{loc}} \) on Majorana chain

\[ \frac{\hbar}{\tau} \approx \frac{\pi}{2} \lambda^2 B_D T \exp(-\Delta/T) \]
\[ \approx \begin{cases} 
\frac{10.7}{e^2\epsilon_F [\text{eV}]} \frac{m^*}{m_0} \left(\frac{l}{d}\right)^2 T \exp(-\Delta/T) & \text{for } D = 1, \\
\frac{281.6}{e^2} \left(\frac{m^*}{m_0}\right)^2 \frac{l [\text{nm}]}{d [\text{nm}]}^4 T \exp(-\Delta/T) & \text{for } D = 3.
\end{cases} \]

exponential suppression but large prefactor

For typical metallic gates one needs \( \Delta > 20 \text{ T} \) for \( \tau > 1 \mu\text{s} \)
Majorana Fermions in Nanowires

Basic ingredients:
s-wave superconductivity
& spin texture due to:

- Spin orbit interaction (SOI)
- Rotating Zeeman field (synthetic SOI, spin helix, …)
- RKKY interaction
Helical spectrum is crucial

‘spin-orbit field’ along z-axis  
e.g. [110] zincblend or Rashba

\[ H_R = \alpha p_x \sigma_z \]

Rashba (1960)

\[ \epsilon_{\pm}(k) = \frac{k^2}{2m} - \mu \mp \sqrt{\alpha^2 k^2 + \Delta^*} \]

‘Helical spectrum’
Helical spectrum is crucial

‘spin-orbit field’ along z-axis
e.g. [110] zincblend or Rashba

\[ H_R = \alpha p_x \sigma_z \]

Rashba (1960)

SOI measured in quantum dots

InAs nanowire \( \lambda_{SO} \sim 100 \) nm,
Fasth, Fuhrer, Samuelson, Golovach & DL,
PRL 98, 266801 (2007)
Competing gaps: magnetic field vs. superconductivity

\[ \Delta_z > \Delta_s : \text{topological phase with Majorana bound state} \]

Volovik, JETP Lett. 70, 609 (1999)
Sato and Fujimoto, PRB 79, 094504 (2009)
nanowires:
Lutchyn et al., PRL 105, 077001 (2010)
Oreg et al., PRL 105, 177002 (2010)

Experiments:
Kouwenhoven group, Science 336, 1003 (2012)
Xu group, Nano Letters 12, 6414 (2012)
Heiblum group, Nat. Phys. 8, 887 (2012)
Rokhinson group, Nat Phys 8, 795 (2012)
Ando group, PRL 107, 217001 (2011) (topological insulators)

Mourik et al., Science 336, 1003 (2012)
Majorana zero-mode detected via ‘zero-bias peak’

Mourik et al., Science 336, 1003 (2012)

Interpretation: Zero bias peak in dI/dV = Majorana fermion

Similar experiments:
Heiblum group, Nat. Phys. 8, 887 (2012)
Xu group, Nano Letters 12, 6414 (2012)
Rokhinson, Nat Phys 8, 795 (2012)
Marcus group: Churchill et al., PRB 87, 241401(R) (2013); Albrecht et al., Nature 531, 206 (2016)
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Composite Structure of Majorana Wavefunction
Klinovaja and DL, PRB 86, 085408 (2012)

Linearization (strong SOI limit):

\[ \Psi(x) = R_\uparrow + L_\downarrow + L_\uparrow e^{-2i k_{so} x} + R_\downarrow e^{2i k_{so} x} \]

\[ \Rightarrow \text{emergent Dirac theory} \]

Exterior and interior branches are decoupled:

Interior branches:

\[ \mathcal{H}^{(i)} = -i \hbar v_F \sigma_3 \partial_x + \Delta_z \sigma_1 \eta_3 + \Delta_{sc} \sigma_2 \eta_2 \]

\[ \phi^{(i)} = (R_\uparrow, L_\downarrow, R_\uparrow^\dagger, L_\downarrow^\dagger) \]

Exterior branches:

\[ \mathcal{H}^{(e)} = i \hbar v_F \sigma_3 \partial_x + \Delta_{sc} \sigma_2 \eta_2 \]

\[ \phi^{(e)} = (L_\uparrow, R_\downarrow, L_\uparrow^\dagger, R_\downarrow^\dagger) \]
Definition of a Majorana fermion

\[ \gamma = \gamma^\dagger \quad \gamma^2 = 1 \]

Majorana: ‘particle that is its own antiparticle’

Fermion basis:

\[
\Psi = \begin{pmatrix} \Psi_{\uparrow} \\ \Psi_{\downarrow} \\ \Psi^\dagger_{\downarrow} \\ \Psi^\dagger_{\uparrow} \end{pmatrix}
\]

then

Wavefunction:

\[
\Phi_{MF} = \begin{pmatrix} f(x) \\ g(x) \\ f^*(x) \\ g^*(x) \end{pmatrix}
\]

MF operator:

\[ \gamma = \int dx \ \Phi_{MF}(x) \cdot \Psi = \gamma^\dagger \]
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Fermion basis:
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\begin{pmatrix}
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f(x) \\
g(x) \\
f^*(x) \\
g^*(x)
\end{pmatrix}
\]

MF operator:
\[
\gamma = \int dx \, \Phi_{MF}(x) \cdot \Psi = \gamma^\dagger
\]

charge is zero
spin is zero
bound state
Typical Majorana wavefunction

\[ \Phi_M(x) = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} e^{-k^{(i)}x} - \begin{pmatrix} i e^{2ik_{so}x} \\ e^{-2ik_{so}x} \end{pmatrix} e^{-k^{(e)}x} \]

from interior branch  

from exterior branch  

two localization lengths, given by inner and outer gap:

\[ \xi^{(i)} = \alpha / |\Delta_s - \Delta_z| \]
\[ \xi^{(e)} = \alpha / \Delta_s \]

Friedel-like oscillations

two branches contribute equally  

exterior branch dominates  

Klinovaja and Loss, PRB 86, 085408 (2012)
Oscillations of Majorana Splitting

Rainis et al., PRB 87, 024515 (2013)

weak SOI

Note: amplitude of oscillation increases!

Marcus group: Albrecht et al., Nature 531, 206 (2016)
Oscillations of Majorana Splitting

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weak SOI

Note: amplitude of oscillation increases!

Amplitude decreases? Why?
Majorana wire and quantum dot
Marcus group: Deng et al., Science 354, 1557 (2016)
Majorana wire and quantum dot

Marcus group: Deng et al., Science 354, 1557 (2016)

Two issues:
1) gap almost closed when ZBP appears
2) ZBP much higher conductance than bulk
Spin and Charge Signatures of Topological Phases

Szumniak, Chevallier, Loss, and Klinovaja, arXiv:1703.00265

\[ \tilde{S}(E_n) = \frac{\hbar}{2} \sum_{j=1}^{N} \Phi_n^\dagger(j) \vec{\sigma} \Phi_n(j) \]

Quasiparticle spin

\[ Q(E_n) = -|e| \sum_{j=1}^{N} \Phi_n^\dagger(j) \tau_z \Phi_n(j) \]

Quasiparticle charge
Band structure: spin polarization

Inversion of bulk spin around topological gap

\[ S_X(k) \parallel B_X \]

\[ \Delta_Z > \Delta_{sc} \]

\[ \Delta_Z < \Delta_{sc} \]

\[ \Delta_{sc} = 0.02, \quad \alpha = 0.3 \]

\[ \mu = 0, \quad |E_g| = \Delta_i \]

Szumniak, Chevallier, Loss, and Klinovaja, arXiv:1703.00265
Note: SOI is given by material property

need to tune chemical potential inside gap
such that $k_F \approx 2k_{\text{SO}}$

Are there *self-tuning* schemes for MFs? Yes!

The SOI interaction can be gauged `away’!

Braunecker, Japaridze, Klinovaja, and DL, PRB 82, 045127 (2010)

Gauge trafo

\[ \Psi_\sigma = e^{-i\sigma k_{so} x} \tilde{\Psi}_\sigma \]

Only rotating field and no SOI anymore!

\[ \tilde{\mathbf{B}} = \mathbf{B}(\cos(2k_{so}x)e_x - \sin(2k_{so}x)e_y) \]
Effective SOI generated by nanomagnets ➔ Majoranas

Klinovaja, Stano, and DL, PRL 109, 236801 (2012)

rotating B-field ➔ effective SOI

\[ \mathbf{B} = B \left( \cos(2k_{so}x)\mathbf{e}_x - \sin(2k_{so}x)\mathbf{e}_y \right) \]

period of nanomagnets = spin-orbit length ~ 20 nm

Graphene nanoribbon: Klinovaja and DL, PRX 3, 011008 (2013)
Majorana Fermions in Atom Chains

Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Localized moments $I_j$ embedded in quasi 1D superconductor

$s$-wave superconductor

$\lambda_F \gg a$

[Princeton group]

Braunecker and Simon, PRL 111, 147202 (2013)
Vazifeh and Franz, PRL 111, 206802 (2013)
Nadj-Perge, Drozdov, Bernevig, and Yazdani, PRB 88, 020407 (2013)
Pientka, Glazman, and v. Oppen, PRB 89, 180505 (2014)
Majorana Fermions in self-tunable RKKY Systems
Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Localized moments $I_j$ embedded in quasi 1D superconductor

Interaction between localized moment $I_i$ at position $R_i$ and itinerant electron spin $\sigma$

$$\mathcal{H}_{int}(x) = \frac{\beta}{2} \sum_i \vec{I}_i \cdot \vec{\sigma} |\psi(r_{\perp,i})|^2 \delta(x - x_i),$$

($\beta \equiv J$

$\beta \ll E_F \Rightarrow$ can integrate out electrons and obtain...
Majorana Fermions in self-tunable RKKY Systems
Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Localized moments $I_j$ embedded in quasi 1D superconductor

\[ \lambda_F \gg \alpha \]

RKKY interaction between moments $I_j$ given by susceptibility of 1D superconductor:

\[ H^{RKKY} = -\frac{2\beta^2}{A^2} \sum_q \chi_q I_q \cdot I_{-q} \]

\[ I_x \propto \cos(2k_F x) \hat{x} + \sin(2k_F x) \hat{y} \]

i.e. helix at Fermi momentum $2k_F$
Helix has period of Fermi wave length

\[ \text{partial gap } \sim \Delta_m \text{ opens at Fermi level } \mu_F \text{ without tuning!} \]

\[ \Delta_m = \alpha \beta \rho_0 \tilde{I}/2 \]

helical Zeeman field seen by electrons (backaction)
Majorana Fermions in self-tunable RKKY Systems
Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Helix opens partial gap *automatically* at Fermi energy $\mu_F$:

Wire in topological phase if

$$\Delta_s < \Delta_m$$

See also, Braunecker and Simon, PRL 111, 147202 (2013)
Vazifeh and Franz, PRL 111, 206802 (2013)
Majorana Fermions in self-tunable RKKY Systems
Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Localized moments $I_j$ embedded in quasi 1D superconductor

RKKY interaction between moments $I_j$ given by susceptibility of 1D superconductor:

$\lambda_F \gg \alpha$

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<th>Chain</th>
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<td>$\beta$</td>
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<td>$\mu_F$</td>
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<tr>
<td>$T_c$</td>
<td>14 K</td>
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Nuclear spin helix & Majoranas in $^{13}$C nanotubes

Chen-Hsuan Hsu, Stano, Klinovaja, and DL, PRB B 92, 235435 (2015)

antiferromagnetic order of nuclear spins

nuclear spin helix opens gaps at $E_F$ , strongly renormalized by e-e interactions:

add superconductivity $\rightarrow$ topological regime with 2 Majoranas at each end of the nanotube

$\Delta_{m} = \tilde{\Delta}_{a} \left( \frac{I A_0}{\Delta_{a}} \right)^{\frac{1}{2-K}}$

$A_{\text{exp}} \sim 100 \mu eV , T_0 \sim 100 \text{ mK}, \Delta_{s} \sim 2 \text{ K}, \xi_{\text{MF}} \sim 3 \mu$m
Majorana Fermions in self-tunable RKKY Systems
Klinovaja, Stano, Yazdani, and DL, PRL 111, 186805 (2013)

Localized moments $I_j$ embedded in quasi 1D superconductor

RKKY interaction between moments $I_j$ given by susceptibility of 1D superconductor:

$$\lambda_F \gg \alpha$$

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Mono-Atomic Fe chains on Superconducting Pb-Surface: Majorana signature

Pawlak et al., npj Quantum Information (2016) 2, 16035
Front-Runners for Quantum Computers

- spin qubits in semiconductors 'small & fast'
- topological quantum computing?
  ‘semi-super devices’
- Majorana Para- or Fibonacci fermions?

semiconducting nanostructures

‘topological spintronics’
Braiding of Majoranas for Topological QC

Kitaev 2003

- topologically protected: Hadamard and $\pi/4$ gate (Clifford gates)
- noisy: CNOT (entangling) and $\pi/8$ gate (non-Clifford)

Universal QC with Majoranas via *noisy gates*, Bravyi, PRA 2008; Landau et al., PRL 2016; Vijay & Fu, arXiv:1609.00950; Karzig et al., arXiv:1610.05289 (Station Q)

Are there alternatives (non-braiding) to generate CNOT and $\pi/8$ gates?
Universal Quantum Computation with Hybrid Spin-Majorana Qubits


Majorana Spin hybrid (`MaSh’) qubit

|0⟩ |1⟩

|0⟩ |1⟩

delocalized fermion:

\[ f_\nu = \frac{\gamma_\nu' + i\gamma_\nu}{2} \]

\[
\begin{cases} 
0 & \text{degenerate states} \\
1 & \text{‘topological protection’}
\end{cases}
\]

But: parity fluctuates at MHz-rate ➔ ‘Majorana box qubit’

Rainis & DL, PRB 85, 174533 (2012); Marcus et al., arxiv:1612.05748
Universal Quantum Computation with Hybrid Spin-Majorana Qubits


Majorana Spin hybrid (’MaSh’) qubit

![Diagram of Majorana Spin hybrid qubit]

- **Spin qubit**
- **odd parity MF qubit**

**delocalized fermion:**

\[ f_\nu = \frac{\gamma_\nu + i\gamma_\nu}{2} \]

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

degenerate states
‘topological protection’

perform perturbation expansion in tunneling t between dot and wires…
Majorana Spin hybrid (‘MaSh’) qubit

\[ \mathcal{H}_{hCP} = 2|t|^2 (1 + \sigma_1)(1 + \eta_1)/\epsilon_0 \]

coherent ‘quantum state transfer’ (faster than projective measurement)

but gate is noisy \( \Rightarrow \) need to control it down to < 1% for surface code
Universal set of gates for Majorana Qubits via Spin Qubit

Hoffman et al., PRB 94, 045316 (2016)
2D Surface code built from topological T-junctions


Majorana Spin hybrid (MaSh) qubit = T-junction nanowire with spin $\frac{1}{2}$ dot

spin rotation ($\pi/8$) (non-Clifford gate)

noisy

qubit-qubit coupling via floating gates and spin exchange $J$ (CNOT)

$$
\mathcal{H}_{MQ}^{(12)} = \frac{|t|^4}{\hbar^2} \mathcal{J} \left[ \sigma_2^{(1)} \sigma_2^{(2)} + \sigma_3^{(1)} \sigma_3^{(2)} \right] \left[ 1 - \eta_1^{(1)} \eta_1^{(2)} \right]
$$

braiding ($H$ & $\pi/4$), topological
What is beyond Majorana femions?
`Parafermion’

(= fractional Majorana fermion zero-energy bound state)

\[\gamma^n = 1\]

\[\gamma \gamma' = \gamma' \gamma e^{-2i\pi/n}, \quad n=2,3,\ldots\]

\[[H, \gamma] = 0 \quad \text{zero mode}\]

Majorana (n=2): can get 2 out of 4 universal quantum gates by braiding
Parafermion (n>2): 2 & weak entanglement*. But: CNOT? (no phase gate)
Fibonacci anyon: 4 universal

Parafermions from QHE edge states hybrid between QHE and SC


A. Vaezi, PRX 4, 031009 (2014)
Double Rashba wire + superconductor


unequal SOI  no B-field !
Double Rashba wire + superconductor


unequal SOI

no B-field!

‘Cooper pair splitter’ physics

Hofstetter et al., Nat. 461, 960 (2009)
Das et al., Nat. Comm. 3, 1165 (2012)
Deacon et al., Nat. Comm. 6, 7446 (2015)
Crossed Andreev Pairing vs. Direct Pairing

C. Reeg, J. Klinovaja and D. Loss, arXiv:1701.07107

how to distinguish between direct and crossed Andreev contributions?

- Crossed Andreev pairing: spin-polarized wires (quantum Hall edge states)

Direct $\propto t_i^2 \propto \gamma_i$

Crossed Andreev $\propto t_L t_R \propto \sqrt{\gamma_L \gamma_R}$

- Direct pairing: decouple one wire
Interplay Between Direct and Crossed Andreev

C. Reeg, J. Klinovaja and D. Loss, arXiv:1701.07107

Comparing three gaps (weak coupling):

\[ E_g(d) = \frac{\gamma \sinh(d/\xi_s)}{\cosh(d/\xi_s)} + |\cos k_F d| \]

\[ E_g^D(d) = \frac{\gamma \sinh(2d/\xi_s)}{\cosh(2d/\xi_s) - \cos 2k_F d} \]

\[ E_g^C(d) = \frac{2\gamma \sinh(d/\xi_s)}{\cosh(2d/\xi_s) - \cos 2k_F d} |\cos k_F d| \]

\[ E_g(d) = E_g^D(d) - E_g^C(d) \]

Direct and crossed Andreev pairing interfere destructively

Proximity-induced gap is minimized when pairing is maximized

Caveat: the standard method of integrating-out SC gives wrong results in general!
Consider inter- and intra-wire pairing

\[ \Delta_c \] ‘crossed Andreev’ term

\[ \Delta_1 \] usual Andreev term

\[ \Delta_\bar{1} \]

**Two competing gap mechanisms**

\[ \Delta_c^2 > \Delta_1 \Delta_\bar{1} \]

due to interactions


- Kramers pairs of Majoranas in crossed Andreev dominated regime
Parafermions: need interactions!

Klinovaja and DL, PRB 90, 045118 (2014)

unequal SOI
no B-field!

Consider inter- and intra-wire pairing \textit{plus} e-e interactions ($g_B$)

‘crossed Andreev’ term $\Delta_c$ usual $\Delta_t$

\[
\mu_{1/3} = E_{so}/3^2
\]

4 Fermi points: $\pm k_{so}(1 \pm 1/3)$
Parafermions: need interactions!

Consider inter- and intra-wire pairing *plus* e-e interactions ($g_B$)

'crossed Andreev' term $\Delta_c$  
usual $\Delta_t$

4 Fermi points: $\pm k_{so}(1 \pm 1/3)$

Note: gap opens only due to interactions

- fractionalized excitations
- parafermions at boundaries

Klinovaja and DL, PRB 90, 045118 (2014)
Topological Stability of Parafermion Phase?

- Parafermions in double wire emerge only due to strong interactions; without interactions the system is gapless.

- Topological stability of ground state...?
  Klinovaja, Mandal, Simon, DL

- Fundamentally different case: start from system with pairing gap $\Delta$ and then add weak interactions $g$ such that gap does not close ($g<<\Delta$) → only topological phases with Majorana fermions are allowed

  Fidowsky & Kitaev 2010/11
  Turner, Berg & Pollmann 2011
Two $Z_3$-Parafermions

3-fold degeneracy (fractional ‘charges’): 0, 2/3, 4/3 (mod 3)

Note: assume that only the spin-up Kramers partner can be occupied (and the spin-down Kramers partner is always empty)
Two $\mathbb{Z}_3$-Parafermions

3-fold degeneracy (fractional ‘charges’): 0, 2/3, 4/3 (mod 2)

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Two $\mathbb{Z}_3$-Parafermions

3-fold degeneracy (fractional ‘charges’): 0, 2/3, 4/3 (mod 2)

use the 3 degenerate states as a qu-trit (3-level system)

4 PFs = 1 qutrit: $|0\>_{L} = |0;0\>$, $|1\>_{L} = |2/3;4/3\>$, $|2\>_{L} = |4/3;2/3\>$
Kramers Pairs of Majorana Fermions and Parafermions in Fractional Topological Insulators

Klinovaja, Yacoby, and DL, PRB 90, 155447 (2014)

Bound states at interfaces between regions of dominant crossed Andreev and usual superconducting pairing terms
Braiding of $Z_d$ Parafermions for TQC

Hutter and DL, PRB 93, 125105 (2016)

**Theorem 2.** If $d$ is odd, braiding of $Z_d$ parafermions allows one to generate the entire many-qudit Clifford group (up to phases).

(missing non-Clifford T-gate for $d$ odd well-studied)

$4$ PFs $= 1$ qudit

$e.g.$ $d=3$ in parity-0 sector, the logical qutrit becomes:

$$\begin{align*}
|0\rangle_L &= |0; 0\rangle, \quad |1\rangle_L = |2/3; 4/3\rangle, \quad |2\rangle_L = |4/3; 2/3\rangle
\end{align*}$$

For $d=3$ we need $S^2$, i.e. $2 \times 12 = 24$ braidings for one CNOT gate

[See also proof by PF diagrammatics, Jaffe, Liu, and Wozniakowski, arXiv:1602.02671]
Summary

- Topological quantum computing
- Majorana fermions in ‘super-semi’ nanowires and chains
- Majorana and spin qubits $\rightarrow$ 2D surface code
- Next generation: Parafermions in double nanowires $\rightarrow$ can get nearly universal TQC including CNOT

Klinovaja and DL, PRB 90, 045118 (2014)  
Hutter and DL, PRB 93, 125105 (2016)