Introduction to the Keldysh Formalism

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0.1 Introduction

These lecture notes are devoted to introduction to Keldysh formalism for treatment of out of equilibrium interacting many–body systems. The name of the technique takes its origin from the 1964 paper of L. V. Keldysh [1]. Among earlier approaches that are closely related to the Keldysh technique, one should mention Konstantinov and Perel [2], Schwinger [3], Kadanoff and Baym [4], and Feynman and Vernon [5]. Classical counterparts of the Keldysh technique are extremely useful and interesting on their own right. Among them Wild diagrammatic technique [6], and Matrin–Siggia–Rose method [7] for stochastic systems (see also related work of DeDominicis [8]).

There is a number of presentations of the method in the existing literature [9, 10, 11, 12, 13, 14, 15]. The emphasis of this review, which is a substantially extended version of Les Houches Session *LXXXI* lectures [16, 17], is on the functional integration approach. It makes the structure and the internal logic of the theory substantially more clear and transparent. We will focus on various applications of the method, exposing connections to other techniques such as the equilibrium Matsubara method [18, 19] and the classical Langevin and Fokker–Planck equations [20, 21]. The major part of the review is devoted to a detailed derivation of the nonlinear σ–model (NLSM) [22, 23, 24, 25], which is probably the most powerful calculation technique in theory of disordered metals and superconductors. This part may be considered as a complimentary material to earlier presentations of the replica [27, 28, 29, 30, 31] and the supersymmetric [32, 33, **?**] versions of the σ–model.

The applications and advantages of Keldysh formulation of the many–body theory among others include:

- Treatment of systems away from thermal equilibrium, either due to the presence of external fields, or in a transient regime.
- An alternative to replica and supersymmetry methods in the theory of systems with quenched disorder.
- Calculation of the full counting statistics of a quantum observable, as opposed to its average value or correlators.
- Treatment of equilibrium problems, where Matsubara analytical continuation may prove to be cumbersome.

0.2 Closed time contour

Consider a quantum many–body system governed by a (possibly time–dependent) Hamiltonian $\hat{H}(t)$. Let us assume that in the distant past

$t = -\infty$ the system was in a state, specified by a many–body density matrix $\hat{\rho}(-\infty)$. The precise form of the latter is of no importance. It may be e.g. the equilibrium density matrix associated with the Hamiltonian $\hat{H}(-\infty)$. The density matrix evolves according to the Heisenberg equation of motion $\partial_t \hat{\rho}(t) = -i\big[\hat{H}(t), \hat{\rho}(t)\big]$, where we set $\hbar = 1$. It is formally solved by $\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big[\hat{\mathcal{U}}_{t,-\infty}\big]^\dagger = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}_{-\infty,t}$, where the evolution operator is given by the time–ordered exponent:

$$\hat{\mathcal{U}}_{t,t'} = \mathbb{T}\exp\left(-i\int_{t'}^{t}\hat{H}(\tau)\mathrm{d}\tau\right) = \lim_{N\to\infty} e^{-i\hat{H}(t)\delta_t}e^{-i\hat{H}(t-\delta_t)\delta_t}\ldots e^{-i\hat{H}(t'+\delta_t)\delta_t}, \tag{0.1}$$

where an infinitesimal time-step is $\delta_t = (t-t')/N$.

One is usually interested in calculations of expectation value for some observable $\hat{\mathcal{O}}$ (say density or current) at a time t, defined as

$$\big\langle\hat{\mathcal{O}}(t)\big\rangle \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}}\mathrm{Tr}\big\{\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big\}\,, \tag{0.2}$$

where the traces are performed over the many–body Hilbert space. The expression under the last trace describes (read from right to left) evolution from $t = -\infty$, where the initial density matrix is specified, forward to t, where the observable is calculated and then backward to $t = -\infty$. Such forward–backward evolution is avoided in the equilibrium by a specially designed trick.

Let us recall e.g. how it works in the zero temperature quantum field theory [19]. The latter deals with the expectation values of the type $\langle \mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle = \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle$, where $|\mathrm{GS}\rangle = \hat{\mathcal{U}}_{t,-\infty}|0\rangle$ is a ground–state of full interacting system. The evolution operator describes the evolution of a simple noninteracting ground state $|0\rangle$ toward $|\mathrm{GS}\rangle$ upon adiabatic switching of the interactions. Now comes the trick: one inserts the operator $\hat{\mathcal{U}}_{+\infty,-\infty}$ in the left most position to accomplish the evolution along the entire time axis. It is then argued that $\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty} = \langle 0|e^{iL}$. This argument is based on the assumption that the system adiabatically follows its ground–state upon slow switching of the interactions "on" and "off" in the distant past and future, correspondingly. Therefore, the only result of evolving the noninteracting ground–state along the entire time axis is acquiring a phase factor e^{iL}. One can then compensate for the added evolution segment by dividing this factor out. As the result: $\langle \mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle = \langle 0|\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}|0\rangle/e^{iL}$ and one faces description of the evolution along the forward time axis without the backward segment. It comes with the price, though: one has to take care of the

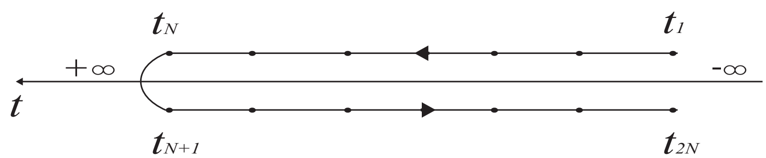

Fig. 0.1. The closed time contour $\mathcal{C}$. Dots on the forward and the backward branches of the contour denote discrete time points.

denominator (which amounts to subtracting of the so–called disconnected diagrams).

Such a trick does not work in a nonequilibrium situation. If the system was driven out of equilibrium, then the final state of its evolution does not have to coincide with the initial one. In general, such a final state depends on the peculiarities of the switching procedure as well as on the entire history of the system. Thus, one can not get rid of the backward portion of the evolution history contained in Eq. (0.2). Schwinger [3] was the first who realized that this is not an unsurmountable obstacle. One has to accept that the evolution in the nonequilibrium quantum field theory takes place along the closed time contour. Along with the conventional forward path, the latter contains the backward one. This way one avoids the need to know the state of the system at $t = +\infty$.

It is still convenient to extend the evolution in Eq. (0.2) to $t = +\infty$ and back to t. It is important to mention that this operation is identical and does not require any additional assumptions. Inserting $\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}$ to the left of $\hat{\mathcal{O}}$ in Eq. (0.2), one obtains

$$\big\langle \hat{\mathcal{O}}(t) \big\rangle = \frac{1}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}} \mathrm{Tr}\big\{\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\big\}\,. \tag{0.3}$$

Here we also used that according to the Heisenberg equation of motion the trace of the density matrix is unchanged under the unitary evolution. As a result, we have obtained the evolution along the closed time contour $\mathcal{C}$ depicted in Fig. 0.2.

The observable $\hat{\mathcal{O}}$ is inserted at time t, somewhere along the forward branch of the contour. Notice that, inserting the unit operator $\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}$ to the right of $\hat{\mathcal{O}}$, we could equally well arrange to have an observable on the backward branch of the contour. As we shall see later, the most convenient choice is to take a half–sum of these two equivalent representations. The observable may be also generated by adding to the Hamiltonian a source term $\hat{H}_{\mathcal{O}}(t) \equiv \hat{H}(t) \pm \hat{\mathcal{O}}\eta(t)/2$, where the plus (minus) signs refer to the forward (backward) parts of the contour. One needs to calculate then the generating functional $Z[\eta]$ defined as the trace of the evolution operator along the contour $\mathcal{C}$ with the Hamiltonian $\hat{H}_{\mathcal{O}}(t)$. Since the latter is non–symmetric on the two branches, such a closed contour evolution operator is not identical to unity. The expectation value of the observable may be then generated as the result of functional differentiation $\langle \hat{\mathcal{O}}(t) \rangle = \delta Z[\eta]/\delta\eta(t)|_{\eta=0}$. We shall first omit the source term and develop a convenient representation for the partition function

$$Z[0] \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}(-\infty)\}}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}} = 1\,, \tag{0.4}$$

where $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,-\infty} = \hat{1}$. The source term, breaking the forward–backward symmetry, will be discussed at a later stage. Notice that since $Z[0] = 1$, the observable may be equally well written in the form, more familiar from the equilibrium context: $\langle \hat{\mathcal{O}}(t) \rangle = \delta \ln Z[\eta]/\delta\eta(t)|_{\eta=0}$. The logarithm is *optional* in the theory with the closed time contour.

The need to carry the evolution along the two–branch contour complicates the nonequilibrium theory in comparison with the equilibrium one. The difficulties may be substantially reduced by a proper choice of variables based on the forward–backward symmetry of the theory. There are also good news: there is no denominator e^{iL}, unavoidably present in the single–branch contour theory. (One should not worry about $\mathrm{Tr}\{\hat{\rho}(-\infty)\}$ in Eq. (0.4). Indeed, this quantity refers entirely to $t = -\infty$, before the interactions were adiabatically switched "on". As a result, it is trivially calculated and never represents a problem.) The absence of the denominator dramatically simplifies description of systems with the quenched disorder. It is the denominator, e^{iL}, which is the main obstacle in performing the disorder averaging of the expectation values of observables. To overcome this obstacle the replica [26, 27, 28] and the supersymmetry [32, 33] tricks were invented. In the closed time contour theory the denominator is absent and thus there is no need in any of these tricks.

0.3 Bosonic coherent states

An extremely useful tool for our purposes is the algebra of bosonic coherent states [34], which we summarize briefly in this paragraph. Consider the bosonic annihilation and creation operators, $\hat{b}$ and $\hat{b}^\dagger$, which operate in the space of the boson occupation numbers n in the following way

$$\hat{b}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,; \qquad \hat{b}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{0.5}$$

The number states $|n\rangle$ form a complete orthonormal basis: $\langle n|n'\rangle = \delta_{nn'}$ and $\sum_n |n\rangle\langle n| = \hat{1}$. By acting on an arbitrary basis state, one may check the following relations

$$\hat{b}^\dagger \hat{b}|n\rangle = n|n\rangle\,; \qquad \hat{b}\hat{b}^\dagger|n\rangle = (n+1)|n\rangle\,; \qquad [\hat{b}, \hat{b}^\dagger] = \hat{1}\,. \tag{0.6}$$

Coherent state, parameterized by a complex number ϕ, is defined as a right eigenstates of the annihilation operator with the eigenvalue ϕ

$$\hat{b}\,|\phi\rangle = \phi|\phi\rangle\,; \qquad \langle\phi|\,\hat{b}^\dagger = \bar{\phi}\langle\phi|\,, \tag{0.7}$$

where the bar denotes complex conjugation. As a result, the matrix elements in the coherent state basis of any *normally ordered* operator $\hat{H}(\hat{b}^\dagger, \hat{b})$ (i.e. all the creation operators are to the left of all the annihilation operators) are given by

$$\langle\phi|\hat{H}(\hat{b}^\dagger, \hat{b})|\phi'\rangle = H(\bar{\phi}, \phi')\,\langle\phi|\phi'\rangle\,. \tag{0.8}$$

One may check by the direct substitution that the following linear superposition of the pure number states is indeed the required right eigenstate of the operator $\hat{b}$:

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}}\,|n\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{n!}\,\big(\hat{b}^\dagger\big)^n|0\rangle = e^{\,\phi\,\hat{b}^\dagger}|0\rangle\,, \tag{0.9}$$

where $|0\rangle$ is the vacuum state, $\hat{b}\,|0\rangle = 0$. Upon Hermitian conjugation, one finds $\langle\phi| = \langle 0|\,e^{\,\bar{\phi}\,\hat{b}} = \sum_n \langle n|\bar{\phi}^n/\sqrt{n!}\,$.

The coherent states are not mutually orthogonal: their set forms an overcomplete basis. The overlap of two coherent states is given by

$$\langle\phi|\phi'\rangle = \sum_{n,n'=0}^{\infty} \frac{\bar{\phi}^n {\phi'}^{n'}}{\sqrt{n!n'!}}\,\langle n|n'\rangle = \sum_{n=0}^{\infty} \frac{(\bar{\phi}\phi')^n}{n!} = e^{\bar{\phi}\phi'}\,, \tag{0.10}$$

where we employed the orthonormality of the pure number states. One may express resolution of unity in the coherent states basis. It takes the following form:

$$\hat{1} = \int \mathrm{d}[\bar{\phi}, \phi]\, e^{-|\phi|^2}\, |\phi\rangle\langle\phi|\,, \tag{0.11}$$

where $\mathrm{d}[\bar{\phi}, \phi] \equiv \mathrm{d}(\mathrm{Re}\,\phi)\,\mathrm{d}(\mathrm{Im}\,\phi)/\pi$. To prove this relation one may employ the Gaussian integral

$$Z[\bar{J}, J] = \int \mathrm{d}[\bar{\phi}, \phi]\, e^{-\bar{\phi}\phi + \bar{\phi}J + \bar{J}\phi} = e^{\,\bar{J}J}\,. \tag{0.12}$$

As its consequence one obtains

$$\int \mathrm{d}[\bar{\phi},\phi]\, e^{-|\phi|^2}\, \bar{\phi}^n \phi^{n'} = \left.\frac{\partial^{n+n'} Z[\bar{J},J]}{\partial J^n \partial \bar{J}^{n'}}\right|_{\bar{J}=J=0} = n!\, \delta_{n,n'} \,. \tag{0.13}$$

Substituting Eq. (0.9) and its conjugate into the right hand side of Eq. (0.11) and employing Eq. (0.13) along with the resolution of unity in the number state basis $\hat{1} = \sum_n |n\rangle\langle n|$, one proves the identity (0.11).

The trace of an arbitrary operator $\hat{O}$, acting in the space of the occupation numbers, is evaluated as

$$\begin{aligned} \mathrm{Tr}\{\hat{O}\} &\equiv \sum_{n=0}^{\infty} \langle n|\hat{O}|n\rangle = \sum_{n=0}^{\infty} \int \mathrm{d}[\bar{\phi},\phi]\, e^{-|\phi|^2}\, \langle n|\hat{O}|\phi\rangle\langle\phi|n\rangle \\ &= \int \mathrm{d}[\bar{\phi},\phi] e^{-|\phi|^2} \sum_{n=0}^{\infty} \langle\phi|n\rangle\langle n|\hat{O}|\phi\rangle = \int \mathrm{d}[\bar{\phi},\phi]\, e^{-|\phi|^2}\, \langle\phi|\hat{O}|\phi\rangle \,, \end{aligned} \tag{0.14}$$

where we have employed resolution of unity first in the coherent state basis and second in the number state basis.

Another useful identity is

$$f(\rho) \equiv \langle\phi|\rho^{\hat{b}^\dagger\hat{b}}|\phi'\rangle = e^{\bar{\phi}\phi'\rho} \,. \tag{0.15}$$

The proof is based on the following operator relation $g(\hat{b}^\dagger\hat{b})\,\hat{b} = \hat{b}\, g(\hat{b}^\dagger\hat{b} - 1)$ valid for an arbitrary function $g(\hat{b}^\dagger\hat{b})$, which is verified by acting on an arbitrary basis vector $|n\rangle$. As a result,

$$\partial_\rho f(\rho) = \langle\phi|\hat{b}^\dagger\hat{b}\, \rho^{\hat{b}^\dagger\hat{b}-1}|\phi'\rangle = \langle\phi|\hat{b}^\dagger \rho^{\hat{b}^\dagger\hat{b}}\, \hat{b}|\phi'\rangle = \bar{\phi}\phi' f(\rho) \,.$$

Integrating this differential equation with the initial condition $f(1) = e^{\bar{\phi}\phi'}$, which follows from Eq. (0.10), one proves the identity (0.15).

0.4 Partition function

Let us consider the simplest many–body system: bosonic particles occupying a single quantum state with the energy ω_0. Its secondary quantized Hamiltonian has the form

$$\hat{H} = \omega_0\, \hat{b}^\dagger\hat{b} \,, \tag{0.16}$$

where $\hat{b}^\dagger$ and $\hat{b}$ are bosonic creation and annihilation operators with the commutation relation $[\hat{b},\hat{b}^\dagger] = \hat{1}$. Let us define the partition function as

$$Z = \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}\}}{\mathrm{Tr}\{\hat{\rho}\}} \,. \tag{0.17}$$

If one assumes that all external fields are exactly the same on the forward and backward branches of the contour, then $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{1}$ and therefore $Z = 1$. The initial density matrix $\hat{\rho} = \hat{\rho}(\hat{H})$ is some operator–valued function of the Hamiltonian. To simplify the derivations one may choose it to be the equilibrium density matrix, $\hat{\rho}_0 = \exp\{-\beta(\hat{H} - \mu\hat{N})\} = \exp\{-\beta(\omega_0 - \mu)\hat{b}^\dagger\hat{b}\}$, where $\beta = 1/T$ is the inverse temperature and μ is the chemical potential. Since arbitrary external perturbations may be switched on (and off) at a later time, the choice of the equilibrium initial density matrix does not prevent one from treating nonequilibrium dynamics. For the equilibrium initial density matrix one finds

$$\mathrm{Tr}\{\hat{\rho}_0\} = \sum_{n=0}^{\infty} e^{-\beta(\omega_0-\mu)n} = [1 - \rho(\omega_0)]^{-1}\,, \tag{0.18}$$

where $\rho(\omega_0) = e^{-\beta(\omega_0-\mu)}$. An important point is that, in general, $\mathrm{Tr}\{\hat{\rho}\}$ is an interaction and disorder independent constant. Indeed, both interactions and disorder are switched on (and off) on the forward (backward) parts of the contour sometime after (before) $t = -\infty$. This constant is therefore frequently omitted without causing confusion.

The next step is to divide the $\mathcal{C}$ contour into $(2N - 2)$ time intervals of length δ_t, such that $t_1 = t_{2N} = -\infty$ and $t_N = t_{N+1} = +\infty$ as shown in Fig. 0.2. One then inserts the resolution of unity in the over–complete coherent state basis, Eq. (0.11),

$$\hat{1} = \int \mathrm{d}[\bar{\phi}_j, \phi_j]\, e^{-|\phi_j|^2} |\phi_j\rangle\langle\phi_j| \tag{0.19}$$

at each point $j = 1, 2, \ldots, 2N$ along the contour. For example, for $N = 3$ one obtains the following sequence in the expression for $\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_0\}$, see Eq. (0.14) (read from right to left):

$$\langle\phi_6|\hat{\mathcal{U}}_{-\delta_t}|\phi_5\rangle\langle\phi_5|\hat{\mathcal{U}}_{-\delta_t}|\phi_4\rangle\langle\phi_4|\hat{1}|\phi_3\rangle\langle\phi_3|\hat{\mathcal{U}}_{+\delta_t}|\phi_2\rangle\langle\phi_2|\hat{\mathcal{U}}_{+\delta_t}|\phi_1\rangle\langle\phi_1|\hat{\rho}_0|\phi_6\rangle\,, \tag{0.20}$$

where $\hat{\mathcal{U}}_{\pm\delta_t}$ is the evolution operator during the time interval δ_t in the positive (negative) time direction. Its matrix elements are given by:

$$\begin{aligned}\left\langle\phi_j\left|\hat{\mathcal{U}}_{\pm\delta_t}\right|\phi_{j-1}\right\rangle &\equiv \left\langle\phi_j\left|e^{\mp i\hat{H}(b^\dagger,b)\delta_t}\right|\phi_{j-1}\right\rangle \approx \left\langle\phi_j\left|\left(1 \mp i\hat{H}(b^\dagger,b)\delta_t\right|\phi_{j-1}\right.\right\rangle \\ &= \langle\phi_j|\phi_{j-1}\rangle\big(1 \mp iH(\bar{\phi}_j,\phi_{j-1})\delta_t\big) \approx e^{\bar{\phi}_j\phi_{j-1}}\, e^{\mp iH(\bar{\phi}_j,\phi_{j-1})\delta_t}\,,\end{aligned} \tag{0.21}$$

where the approximate equalities are valid up to the linear order in δ_t. Here we have employed expression (0.8) for the matrix elements of a normally-ordered operator along with Eq. (0.10) for the overlap of the coherent states. Equation (0.21) is not restricted to the toy example (0.16), but holds for any

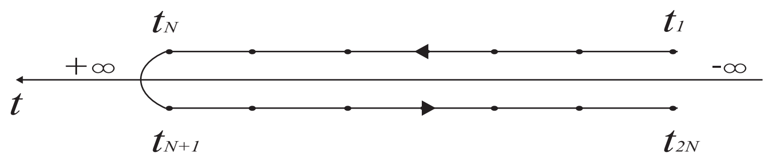

Fig. 0.2. The closed time contour $\mathcal{C}$. Dots on the forward and the backward branches of the contour denote discrete time points.

normally–ordered Hamiltonian. Notice that there is no evolution operator inserted between t_N and t_{N+1}. Indeed, these two points are physically indistinguishable and thus the system does not evolve during this time interval.

Employing the following property of coherent states $\langle\phi|e^{-\beta(\omega_0-\mu)b^\dagger b}|\phi'\rangle = \exp\left\{\bar{\phi}\phi'\rho(\omega_0)\right\}$ (cf. Eq. (0.15)) and collecting all the exponential factors along the contour, one finds for the partition function, Eq. (0.17),

$$Z = \frac{1}{\mathrm{Tr}\{\hat{\rho}_0\}} \int \prod_{k=1}^{2N} \mathrm{d}[\bar{\phi}_k, \phi_k] \, \exp\left(i \sum_{j,j'=1}^{2N} \bar{\phi}_j \, G^{-1}_{jj'} \, \phi_{j'} \right) . \tag{0.22}$$

For $N = 3$ the $2N \times 2N$ matrix $iG^{-1}_{jj'}$ takes the form

$$iG^{-1}_{jj'} \equiv \left[\begin{array}{ccc|ccc} -1 & & & & & \rho(\omega_0) \\ h_- & -1 & & & & \\ & h_- & -1 & & & \\ \hline & & 1 & -1 & & \\ & & & h_+ & -1 & \\ & & & & h_+ & -1 \end{array}\right] , \tag{0.23}$$

where $h_\mp \equiv 1 \mp i\omega_0\delta_t$. The main diagonal of this matrix originates from the resolution of unity, Eq. (0.19), while the lower sub-diagonal from the matrix elements (0.21). Finally the upper right element comes from $\langle\phi_1|\hat{\rho}_0|\phi_{2N}\rangle$ in Eq. (0.20). This structure of the $i\hat{G}^{-1}$ matrix is straightforwardly generalized on arbitrary N.

To proceed with the multiple integrals, appearing in Eq. (0.22), we remind the reader of some properties of Gaussian integrals.

0.5 Bosonic Gaussian integrals

For any complex $N \times N$ matrix $\hat{A}_{ij}$, where $i, j = 1, \ldots N$, such that all its eigenvalues, λ_i, have a positive real part, $\mathrm{Re}\lambda_i > 0$, the following statement holds

$$Z[\bar{J},J] = \int \prod_{k=1}^{N} \mathrm{d}[\bar{z}_k, z_k]\; e^{-\sum\limits_{ij}^{N} \bar{z}_i \hat{A}_{ij} z_j + \sum\limits_{j}^{N}\left[\bar{z}_j J_j + \bar{J}_j z_j\right]} = \frac{e^{\sum\limits_{ij}^{N} \bar{J}_i (\hat{A}^{-1})_{ij} J_j}}{\det\hat{A}}\,, \quad (0.24)$$

where J_j is an arbitrary complex vector and $\mathrm{d}[\bar{z}_j z_j] = \mathrm{d}(\mathrm{Re}z_j)\mathrm{d}(\mathrm{Im}z_j)/\pi$. This equality is a generalization of the Gaussian integral (0.12), used above. To prove it, one starts from a Hermitian matrix $\hat{A}$, which may be diagonalized by a unitary transformation: $\hat{A} = \hat{U}^\dagger \hat{\Lambda} \hat{U}$, where $\hat{\Lambda} = \mathrm{diag}\{\lambda_j\}$. The identity is then proven by a change of variables with a unit Jacobian to $w_i = \sum_j \hat{U}_{ij} z_j$, which leads to

$$Z[\bar{J},J] = \prod_{j=1}^{N} \int \mathrm{d}[\bar{w}_j, w_j]\; e^{-\bar{w}_j \lambda_j w_j + \bar{w}_j I_j + \bar{I}_j w_j} = \prod_{j=1}^{N} \frac{e^{\bar{I}_j \lambda_j^{-1} I_j}}{\lambda_j}\,,$$

where $I_i = \sum_j \hat{U}_{ij} J_j$. Using $\sum_j \bar{I}_j \lambda_j^{-1} I_j = \vec{\bar{J}}^T \hat{U}^\dagger \hat{\Lambda}^{-1} \hat{U} \vec{J} = \vec{\bar{J}}^T \hat{A}^{-1} \vec{J}$, along with $\det\hat{A} = \prod_j \lambda_j$, one obtains the right hand side of Eq. (0.24). Finally, one notices that the right hand side of Eq. (0.24) is an analytic function of both $\mathrm{Re}A_{ij}$ and $\mathrm{Im}A_{ij}$. Therefore, one may continue them analytically to the complex plane to reach an arbitrary complex matrix $\hat{A}_{ij}$. The identity (0.24) is thus valid as long as the integral is well defined, that is all the eigenvalues of $\hat{A}$ have a positive real part.

The Wick theorem deals with the average value of $z_{a_1} \ldots z_{a_k} \bar{z}_{b_1} \ldots \bar{z}_{b_k}$ weighted with the factor $\exp\big(-\sum_{ij} \bar{z}_i \hat{A}_{ij} z_j\big)$. The theorem states that this average is given by the sum of all possible products of pair-wise averages. For example,

$$\langle z_a \bar{z}_b \rangle \equiv \frac{1}{Z[0,0]} \frac{\delta^2 Z[\bar{J},J]}{\delta \bar{J}_a \delta J_b}\bigg|_{J=0} = \hat{A}^{-1}_{ab}\,, \quad (0.25)$$

$$\langle z_a z_b \bar{z}_c \bar{z}_d \rangle \equiv \frac{1}{Z[0,0]} \frac{\delta^4 Z[\bar{J},J]}{\delta \bar{J}_a \delta \bar{J}_b \delta J_c \delta J_d}\bigg|_{J=0} = \hat{A}^{-1}_{ac} \hat{A}^{-1}_{bd} + \hat{A}^{-1}_{ad} \hat{A}^{-1}_{bc}\,,$$

etc.

The Gaussian identity for integration over real variables has the form

$$Z[J] = \int \prod_{k=1}^{N} \left(\frac{\mathrm{d}x_k}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}\sum\limits_{ij}^{N} x_i \hat{A}_{ij} x_j + \sum\limits_{j}^{N} x_j J_j} = \frac{e^{\frac{1}{2}\sum\limits_{ij}^{N} J_i (\hat{A}^{-1})_{ij} J_j}}{\sqrt{\det\hat{A}}}\,, \quad (0.26)$$

where $\hat{A}$ is a *symmetric* complex matrix with all its eigenvalues having a positive real part. The proof is similar to those in the case of complex variables: one starts from a real symmetric matrix, which may be diagonalized by an orthogonal transformation. The identity (0.26) is then easily proven by a change of the variables. Finally, one may analytically continue the r.h.s. (as long as the integral is well defined) from a real symmetric matrix $\hat{A}_{ij}$, to a *complex symmetric* one.

The corresponding Wick theorem for the average value of $x_{a_1} \dots x_{a_{2k}}$ weighted with the factor $\exp\big(-\frac{1}{2}\sum_{ij} x_i \hat{A}_{ij} x_j\big)$ takes the form

$$\langle x_a x_b \rangle \equiv \frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J_a \delta J_b}\bigg|_{J=0} = \hat{A}^{-1}_{ab}\,, \tag{0.27}$$

$$\langle x_a x_b x_c x_d \rangle \equiv \frac{1}{Z[0]} \frac{\delta^4 Z[J]}{\delta J_a \delta J_b \delta J_c \delta J_d}\bigg|_{J=0} = \hat{A}^{-1}_{ab}\hat{A}^{-1}_{cd} + \hat{A}^{-1}_{ac}\hat{A}^{-1}_{bd} + \hat{A}^{-1}_{ad}\hat{A}^{-1}_{bc}\,,$$

etc. Notice the additional term in the second line in comparison with the corresponding complex result (0.25). The symmetry of $\hat{A}$ (and thus of $\hat{A}^{-1}$) is necessary to satisfy the obvious relation $\langle x_a x_b \rangle = \langle x_b x_a \rangle$.

0.6 Normalization and continuum notations

Having established Gaussian identity (0.24) one can apply it to Eq. (0.22) to check the normalization factor. In this case $\hat{A} = -i\hat{G}^{-1}$ and it is straightforward to evaluate the corresponding determinant employing Eq. (0.23)

$$\begin{aligned} \det\big[-i\hat{G}^{-1}\big] \;&=\; 1 - \rho(\omega_0)(h_- h_+)^{N-1} = 1 - \rho(\omega_0)\left(1 + \omega_0^2 \delta_t^2\right)^{N-1} \\ &\approx\; 1 - \rho(\omega_0)\, e^{\omega_0^2 \delta_t^2 (N-1)} \overset{N\to\infty}{\longrightarrow} 1 - \rho(\omega_0), \end{aligned} \tag{0.28}$$

where one used that $\delta_t^2 N \to 0$ if $N \to \infty$. Indeed, we divide the contour in a way to keep $\delta_t N = \text{const}$ (given by a full extent of the time axis) as a result $\delta_t^2 \sim N^{-2}$. Employing the fact that the Gaussian integral in Eq. (0.22) is equal to the inverse determinant of $-i\hat{G}^{-1}$ matrix, Eq. (0.24), along with Eq. (0.18), one finds

$$Z = \frac{1}{\mathrm{Tr}\{\hat{\rho}_0\}} \frac{1}{\det\big[-i\hat{G}^{-1}\big]} = 1\,, \tag{0.29}$$

as it should be, of course. Notice that keeping the upper–right element of the discrete matrix, Eq. (0.23), is crucial to maintain this normalization identity.

One may now take the limit $N \to \infty$ and formally write the partition function (0.22) in the continuum notations, $\phi_j \to \phi(t)$, as

$$Z = \int \mathbf{D}[\bar{\phi}(t), \phi(t)]\; e^{iS[\bar{\phi},\phi]}\,, \tag{0.30}$$

where the integration measure is the shorthand notation for $\mathbf{D}[\bar{\phi}(t), \phi(t)] =$

$\prod_{j=1}^{2N} \mathrm{d}[\bar{\phi}_j, \phi_j]/\,\mathrm{Tr}\{\hat{\rho}_0\}$. According to Eqs. (0.22) and (0.23), the action is given by

$$S[\bar{\phi},\phi] = \sum_{j=2}^{2N} \delta t_j \left[i\bar{\phi}_j \frac{\phi_j - \phi_{j-1}}{\delta t_j} - \omega_0 \bar{\phi}_j \phi_{j-1} \right] + i\,\bar{\phi}_1 \Big[\phi_1 - i\rho(\omega_0)\phi_{2N}\Big]\,, \tag{0.31}$$

where $\delta t_j \equiv t_j - t_{j-1} = \pm\delta_t$ on the forward and backward branches, correspondingly. In continuum notations, $\phi_j \to \phi(t)$, the action acquires the form

$$S[\bar{\phi},\phi] = \int_{\mathcal{C}} \mathrm{d}t\, \bar{\phi}(t) \hat{G}^{-1} \phi(t)\,, \tag{0.32}$$

where the continuum form of the operator $\hat{G}^{-1}$ is

$$\hat{G}^{-1} = i\partial_t - \omega_0\,. \tag{0.33}$$

It is extremely important to remember that this continuum notation is only an abbreviation which represents the large discrete matrix, Eq. (0.23). In particular, the upper–right element of the matrix (the last term in Eq. (0.31)), that contains the information about the distribution function, is seemingly absent in the continuum notations Eq. (0.33). The necessity to keep the boundary terms originates from the fact that the continuum operator (0.33) possesses the zero mode $e^{-i\omega_0 t}$. Its inverse operator $\hat{G}$ is therefore not uniquely defined, unless the boundary terms are included.

To avoid integration along the closed time contour it is convenient to split the bosonic field $\phi(t)$ into the two components $\phi^+(t)$ and $\phi^-(t)$ which reside on the forward and the backward parts of the time contour respectively. The continuum action may be then rewritten as

$$S[\bar{\phi},\phi] = \int_{-\infty}^{+\infty} \mathrm{d}t\, \big[\bar{\phi}^+(t)(i\partial_t - \omega_0)\phi^+(t) - \bar{\phi}^-(t)(i\partial_t - \omega_0)\phi^-(t)\big]\,, \tag{0.34}$$

where the relative minus sign comes from the reversed direction of the time integration on the backward part of the contour. Once again, the continuum notations are somewhat misleading. Indeed, they create an undue impression that $\phi^+(t)$ and $\phi^-(t)$ fields are completely uncorrelated. In fact, they are connected due to the presence of the nonzero off–diagonal blocks in the discrete matrix, Eq. (0.23). It is therefore desirable to develop a continuum representation that automatically takes into account the proper regularization and mutual correlations. We shall achieve it in the following sections. First the Green functions should be discussed.

0.7 Green functions

According to the basic properties of the Gaussian integrals, see section 0.5, the correlator of the two bosonic fields is given by

$$\langle \phi_j\, \bar\phi_{j'} \rangle \equiv \int \mathbf{D}[\bar\phi,\phi]\ \phi_j\, \bar\phi_{j'} \exp\left(i \sum_{k,k'=1}^{2N} \bar\phi_k\, G^{-1}_{kk'}\, \phi_{k'} \right) = iG_{jj'}\,. \qquad (0.35)$$

Notice, the absence of the factor Z^{-1} in comparison with the analogous definition in the equilibrium theory [34]. Indeed, in the present construction $Z = 1$. This seemingly minor difference turns out to be the major issue in the theory of disordered systems (see further discussion in chapter **??**, devoted to fermions with the quenched disorder). Inverting the $2N \times 2N$ matrix (0.23) with $N = 3$, one finds

$$iG_{jj'} = \frac{1}{\det\big[-i\hat G^{-1}\big]} \left[\begin{array}{ccc|ccc} 1 & \rho h_+^2 h_- & \rho h_+^2 & \rho h_+^2 & \rho h_+ & \rho \\ h_- & 1 & \rho h_+^2 h_- & \rho h_+^2 h_- & \rho h_+ h_- & \rho h_- \\ h_-^2 & h_- & 1 & \rho h_+^2 h_-^2 & \rho h_+ h_-^2 & \rho h_-^2 \\ \hline h_-^2 & h_- & 1 & 1 & \rho h_-^2 h_+ & \rho h_-^2 \\ h_-^2 h_+ & h_- h_+ & h_+ & h_+ & 1 & \rho h_-^2 h_+ \\ h_-^2 h_+^2 & h_- h_+^2 & h_+^2 & h_+^2 & h_+ & 1 \end{array} \right], \qquad (0.36)$$

where $\rho \equiv \rho(\omega_0)$. Generalization of the $N = 3$ example to an arbitrary N is again straightforward. We switch now to the fields $\phi_j^{\pm}$, residing on the forward (backward) branches of the contour. Hereafter $j = 1, \dots, N$ and thus the $2N \times 2N$ matrix written above is indexed as $1, 2, \dots, N, N, \dots, 2, 1$. Then the following correlations may be read out of the matrix (0.36):

$$\langle \phi_j^+ \bar\phi_{j'}^- \rangle \equiv iG^<_{jj'} = \frac{\rho\, h_+^{j'-1} h_-^{j-1}}{\det\big[-i\hat G^{-1}\big]}\,, \qquad (0.37a)$$

$$\langle \phi_j^- \bar\phi_{j'}^+ \rangle \equiv iG^>_{jj'} = \frac{h_+^{N-j} h_-^{N-j'}}{\det\big[-i\hat G^{-1}\big]} = \frac{(h_+h_-)^{N-1} h_+^{1-j} h_-^{1-j'}}{\det\big[-i\hat G^{-1}\big]}\,, \qquad (0.37b)$$

$$\langle \phi_j^+ \bar\phi_{j'}^+ \rangle \equiv iG^{\mathbb{T}}_{jj'} = \frac{h_-^{j-j'}}{\det\big[-i\hat G^{-1}\big]} \times \begin{cases} 1 & j \geq j' \\ \rho(h_+h_-)^{N-1} & j < j' \end{cases}, \qquad (0.37c)$$

$$\langle \phi_j^- \bar\phi_{j'}^- \rangle \equiv iG^{\tilde{\mathbb{T}}}_{jj'} = \frac{h_+^{j'-j}}{\det\big[-i\hat G^{-1}\big]} \times \begin{cases} \rho(h_+h_-)^{N-1} & j > j' \\ 1 & j \leq j' \end{cases}. \qquad (0.37d)$$

Here the symbols $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ stay for time ordering and anti-ordering correspondingly, while $<$ $(>)$ is a convenient notation indicating that the first time argument is taken before (after) the second one on the Keldysh contour.

Recalling that $h_\mp = 1 \mp i\omega_0\delta_t$, one can take the $N \to \infty$ limit, keeping $N\delta_t$ a constant. To this end notice that $(h_+h_-)^N = (1+\omega_0^2\delta_t^2)^N \stackrel{N\to\infty}{\longrightarrow} 1$, while $h_\mp^j \stackrel{N\to\infty}{\longrightarrow} e^{\mp i\omega_0\delta_t j} = e^{\mp i\omega_0 t}$, where we denoted $t = \delta_t j$ and correspondingly $t' = \delta_t j\,'$. Employing also the evaluation of the determinant given by Eq. (0.28), one obtains for the correlation functions in the continuum limit

$$\langle\phi^+(t)\bar\phi^-(t')\rangle = iG^<(t,t') = n_B\, e^{-i\omega_0(t-t')}\,, \tag{0.38a}$$

$$\langle\phi^-(t)\bar\phi^+(t')\rangle = iG^>(t,t') = (n_B+1)\, e^{-i\omega_0(t-t')}\,, \tag{0.38b}$$

$$\langle\phi^+(t)\bar\phi^+(t')\rangle \;= iG^{\mathbb{T}}(t,t') = \theta(t-t')iG^>(t,t') + \theta(t'-t)iG^<(t,t')\,, \tag{0.38c}$$

$$\langle\phi^-(t)\bar\phi^-(t')\rangle = iG^{\widetilde{\mathbb{T}}}(t,t') = \theta(t'-t)iG^>(t,t') + \theta(t-t')iG^<(t,t')\,, \tag{0.38d}$$

where we introduced *bosonic occupation number* n_B as

$$n_B(\omega_0) = \frac{\rho(\omega_0)}{1-\rho(\omega_0)}\,. \tag{0.39}$$

Indeed, to calculate number of bosons at a certain point in time one needs to insert the operator $\hat b^\dagger\hat b$ into the corresponding point along the forward or backward branches of the contour. This leads to the correlation function $\langle\phi_{j-1}\bar\phi_j\rangle$, or in terms of $\phi^\pm$ fields to either $\langle\phi^+_{j-1}\bar\phi^+_j\rangle$ or $\langle\phi^-_j\bar\phi^-_{j-1}\rangle$ (notice the reversed indexing along the backward branch). According to Eqs. (0.37c,d) in the $N\to\infty$ limit both of them equal n_B.

The step–function $\theta(t)$ in Eqs. (0.38c,d) is defined as $\theta(t-t') = 1$, if $t > t'$ and $\theta(t-t') = 0$, if $t < t'$. There is an ambiguity about equal times. Consulting with the discrete version of the correlation functions, Eqs. (0.37), one notices that in both equations (0.38c) and (0.38d) the first step function should be understood as having $\theta(0) = 1$, while the second as having $\theta(0) = 0$. Although slightly inconvenient, this ambiguity will disappear in the formalism that follows.

In analogy with the definition of the discrete correlation functions as a $2N$-fold integral, Eq. (0.35), it is convenient to write formally their continuum limit, Eq. (0.38), as a *functional* integral

$$\big\langle\phi^\pm(t)\,\bar\phi^\pm(t')\big\rangle = \int \mathbf{D}[\bar\phi,\phi]\,\phi^\pm(t)\bar\phi^\pm(t')\,e^{\,iS[\bar\phi,\phi]}\,, \tag{0.40}$$

where the action $S[\bar\phi,\phi]$ is given by Eq. (0.34). Notice that, despite the impression that the integrals over $\phi^+(t)$ and $\phi^-(t)$ may be split from each other and performed separately, there are non-vanishing cross-correlations between these fields, Eqs. (0.38a,b). The reason, of course, is that the continuum notation (0.40) is nothing but the shorthand abbreviation for $N \to \infty$ limit of the discrete integral (0.35). The latter contains the matrix (0.23) with non-zero off-diagonal blocks, which are the sole reason for the existence of the cross-correlations. It is highly desirable to develop a continuum formalism, which automatically accounts for the proper cross-correlations without the need to resort to the discrete notations.

This task is facilitated by the observation that not all four Green functions defined above are independent. Indeed, direct inspection shows that

$$G^{\mathbb{T}}(t,t') + G^{\widetilde{\mathbb{T}}}(t,t') - G^{>}(t,t') - G^{<}(t,t') = 0\,. \tag{0.41}$$

This suggests that one may benefit explicitly from this relation by performing a linear transformation. The Keldysh rotation achieves just that. Notice that, due to the regularization of $\theta(0)$ discussed above, the identity does *not* hold for $t = t'$. Indeed at $t = t'$ the left hand side of Eq. (0.41) is one rather than zero. However, since $t = t'$ line is a manifold of measure zero, the violation of Eq. (0.41) for most purposes is inconsequential. (Notice that the right hand side of Eq. (0.41) is not a delta-function $\delta(t-t')$. It is rather a Kronecker delta $\delta_{jj'}$ in the discrete version, which disappears in the continuum limit).

0.8 Keldysh rotation

Let us introduce a new pair of fields according to

$$\phi^{cl}(t) = \frac{1}{\sqrt{2}}\big(\phi^{+}(t) + \phi^{-}(t)\big)\,, \qquad \phi^{q}(t) = \frac{1}{\sqrt{2}}\big(\phi^{+}(t) - \phi^{-}(t)\big)\,, \tag{0.42}$$

with the analogous transformation for the conjugated fields. The superscripts *"cl"* and *"q"* stand for the *classical* and the *quantum* components of the fields correspondingly. The rationale for these notations will become clear shortly. First, a simple algebraic manipulation with Eqs. (0.37a)–(0.37d) shows that

$$\big\langle \phi^{\alpha}(t)\,\bar{\phi}^{\beta}(t')\big\rangle \equiv iG^{\alpha\beta}(t,t') = \begin{pmatrix} iG^{K}(t,t') & iG^{R}(t,t') \\ iG^{A}(t,t') & 0 \end{pmatrix}\,, \tag{0.43}$$

where hereafter $\alpha, \beta = (cl, q)$. The fact that the (q,q) element of this matrix is zero is a manifestation of the identity (0.41). Superscripts R, A and K stand for *retarded, advanced* and *Keldysh* components of the Green function respectively. These three Green functions are the fundamental objects of the Keldysh technique. They are defined as

$$G^R(t,t') = G^{cl,q}(t,t') = \frac{1}{2}\left(G^{\mathbb{T}} - G^{\widetilde{\mathbb{T}}} + G^{>} - G^{<}\right) = \theta(t-t')\left(G^{>} - G^{<}\right) \tag{0.44a}$$

$$G^A(t,t') = G^{q,cl}(t,t') = \frac{1}{2}\left(G^{\mathbb{T}} - G^{\widetilde{\mathbb{T}}} - G^{>} + G^{<}\right) = \theta(t'-t)\left(G^{<} - G^{>}\right) \tag{0.44b}$$

$$G^K(t,t') = G^{cl,cl}(t,t') = \frac{1}{2}\left(G^{\mathbb{T}} + G^{\widetilde{\mathbb{T}}} + G^{>} + G^{<}\right) = G^{>} + G^{<} \tag{0.44c}$$

As was mentioned below Eq. (0.41), these relations hold for $t \neq t'$ only, while the diagonal $t = t'$ is discussed below. Since by definition $[G^{<}]^\dagger = -G^{>}$ [cf. Eq. (0.37)], one notices that

$$G^A = \left[G^R\right]^\dagger, \qquad\qquad G^K = -\left[G^K\right]^\dagger, \tag{0.45}$$

where the Green functions are understood as matrices in the time domain. Hermitian conjugation therefore includes complex conjugation along with interchanging the two time arguments.

The retarded (advanced) Green function is a lower (upper) triangular matrix in the time domain. Since a product of any number of triangular matrices is again a triangular matrix, one obtains the simple rule that the convolution of any number of retarded (advanced) Green functions is also a retarded (advanced) Green function

$$G_1^R \circ G_2^R \circ \ldots \circ G_l^R = G^R\,, \tag{0.46a}$$

$$G_1^A \circ G_2^A \circ \ldots \circ G_l^A = G^A\,, \tag{0.46b}$$

where the circular multiplication sign stands for the convolution operation, i.e. multiplication of matrices in the time domain and subscripts denote all other indexes apart of the time.

Both retarded and advanced matrices have non-zero main diagonal, i.e. $t = t'$. The important observation, however, is that

$$G^R(t,t) + G^A(t,t) = 0\,, \tag{0.47}$$

see Eqs. (0.38c,d) and the discussion of $\theta(0)$ regularization below them. Since in all subsequent calculations it is always the sum of the two which is important, one may use, as a rule of thumb, that both retarded and advanced Green functions separately vanish at coinciding time arguments. Provided the relation (0.47) is always understood, there is no danger in extending

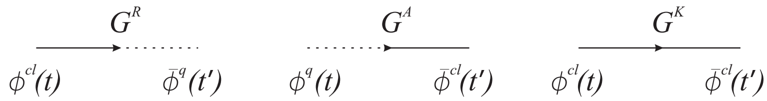

Fig. 0.3. Graphic representation of G^R, G^A, and G^K correspondingly. The full line represents the classical field component ϕ^{cl}, while the dashed line the quantum component ϕ^q.

Eq. (0.44) to the diagonal $t = t'$ in the continuum formalism. In the energy representation Eq. (0.47) takes the form

$$\int \frac{\mathrm{d}\epsilon}{2\pi} \left[G^R(\epsilon) + G^A(\epsilon)\right] = 0\,. \tag{0.48}$$

Once again, although it is only the integral of the sum of the two which vanishes, one may use, as a rule of thumb, that energy integrals of retarded and advanced Green functions are separately zero. This is due to the fact that retarded (advanced) functions are analytic functions in the entire upper (lower) half plane of the complex energy argument. Therefore, by closing the energy integration contour at infinity, one expects the integral to be zero.

It is useful to introduce graphic representations for the three Green functions. To this end, let us denote the classical component of the field by a full line and the quantum component by a dashed line. Then the retarded Green function is represented by a full–arrow–dashed line, the advanced by a dashed–arrow–full line and the Keldysh by full–arrow–full line, see Fig. 0.3. Notice that the dashed–arrow–dashed line, that would represent the $\langle \phi^q \bar{\phi}^q \rangle$ Green function, is absent. The arrow shows the direction from ϕ^α towards $\bar{\phi}^\beta$.

Employing Eqs. (0.38), one finds for our toy example of the single boson level

$$G^R = -i\theta(t-t')\, e^{-i\omega_0(t-t')} \to (\epsilon - \omega_0 + i0)^{-1}\,, \tag{0.49a}$$

$$G^A = \quad i\theta(t'-t)\, e^{-i\omega_0(t-t')} \to (\epsilon - \omega_0 - i0)^{-1}\,, \tag{0.49b}$$

$$G^K = -i\left[2n_B(\omega_0) + 1\right] e^{-i\omega_0(t-t')} \to -2\pi i[2n_B(\epsilon) + 1]\delta(\epsilon - \omega_0)\,. \tag{0.49c}$$

The Fourier transforms with respect to $t - t'$ are given for each of the three Green functions. Notice that the retarded and advanced components contain information only about the spectrum and are independent of the occupation number, whereas the Keldysh component depends on it. In thermal equilibrium $\rho = e^{-(\omega_0-\mu)/T}$, while $n_B = (e^{(\omega_0-\mu)/T}-1)^{-1}$ and therefore

$$G^K(\epsilon) = \coth\frac{\epsilon-\mu}{2T}\,\left[G^R(\epsilon) - G^A(\epsilon)\right] . \tag{0.50}$$

The last equation constitutes the statement of the *fluctuation–dissipation theorem* (FDT). The FDT is, of course, a general property of thermal equilibrium that is not restricted to the toy example, considered here. It implies a rigid relation between the response functions and the correlation functions in equilibrium.

In general, it is convenient to parameterize the anti–Hermitian Keldysh Green function, Eq. (0.45), with the help of a Hermitian matrix $F = F^\dagger$, as follows

$$G^K = G^R \circ F - F \circ G^A \,, \tag{0.51}$$

where $F = F(t,t')$. The Wigner transform (see chapter **??**), $\mathbf{f}(\tau,\epsilon)$, of the matrix F is referred to as the *distribution function.* In thermal equilibrium $\mathbf{f}(\epsilon) = \coth((\epsilon-\mu)/2T)$, Eq. (0.50).

0.9 Keldysh action and its structure

One would like to have a continuum action, written in terms of ϕ^{cl},ϕ^q, that properly reproduces the correlators Eqs. (0.43) and (0.49) i.e.

$$\left\langle \phi^\alpha(t)\,\bar\phi^\beta(t')\right\rangle = iG^{\alpha\beta}(t,t') = \int \mathbf{D}[\phi^{cl},\phi^q]\;\phi^\alpha(t)\,\bar\phi^\beta(t')\;e^{iS[\phi^{cl},\phi^q]}\,, \tag{0.52}$$

where the conjugated fields are not listed in the action arguments and in the integration measure for brevity. According to the basic properties of Gaussian integrals, section 0.5, the action should be taken as a quadratic form of the fields with the matrix which is an inverse of the correlator $G^{\alpha\beta}(t,t')$. Inverting the matrix (0.43), one thus finds the proper action

$$S[\phi^{cl},\phi^q] = \iint_{-\infty}^{+\infty} \mathrm{d}t\,\mathrm{d}t'\,\left(\bar\phi^{cl},\bar\phi^q\right)_t \left(\begin{array}{cc} 0 & \left[G^{-1}\right]^A \\ \left[G^{-1}\right]^R & \left[G^{-1}\right]^K \end{array}\right)_{t,t'} \left(\begin{array}{c}\phi^{cl}\\ \phi^q\end{array}\right)_{t'} . \tag{0.53}$$

The off-diagonal elements are found from the condition $\left[G^{-1}\right]^R \circ G^R = 1$ and the similar one for the advanced component. The right hand side here is the unit matrix, which in the time representation is $\delta(t-t')$. As a result, the off-diagonal components are obtained by the matrix inversion of the corresponding components of the Green functions $\left[G^{-1}\right]^{R(A)} = \left[G^{R(A)}\right]^{-1}$. Such an inversion is most convenient in the energy representation

$$\left[G^{-1}\right]^{R(A)} = \epsilon - \omega_0 \pm i0 \to \delta(t-t')\,(i\partial_{t'} - \omega_0 \pm i0)\;, \tag{0.54a}$$

where in the last step we performed the inverse Fourier transform back to the time representation, employing that the Fourier transform of ϵ is $\delta(t-t')i\partial_{t'}$.

Although in the continuum limit these matrices look diagonal, it is important to remember that in the discrete regularization $\left[G^{R(A)}\right]^{-1}$ contains $\mp i$ along the main diagonal and $\pm i - \omega_0\delta_t$ along the lower(upper) sub-diagonal. The determinants of the corresponding matrices are given by the product of all diagonal elements $\det\left[G^{-1}\right]^R \det\left[G^{-1}\right]^A = \prod_{j=1}^N i(-i) = 1$. To obtain this statement without resorting to discretization, one notices that in the energy representation the Green functions are diagonal and therefore $\det\left[G^{-1}\right]^R\left[G^{-1}\right]^A = \prod_\epsilon\left[G^R(\epsilon)G^A(\epsilon)\right]^{-1} = \exp\left\{-\int\frac{d\epsilon}{2\pi}[\ln G^R + \ln G^A]\right\} = 1$. Here we used the fact that Eq. (0.48) holds not only for the Green functions themselves, but also for any function of them. This property is important for maintaining the normalization identity $Z = \int \mathbf{D}[\phi^{cl},\phi^q]\, e^{iS} = 1$. Indeed, the integral is equal to minus (due to the factor of i in the exponent) the determinant of the quadratic form, while the latter is (-1) times the product of the determinants of the off diagonal elements in the quadratic form (0.53).

The diagonal Keldysh component of the quadratic form (0.53) is found from the condition $G^K\circ[G^A]^{-1} + G^R\circ[G^{-1}]^K = 0$. Employing the parametrization (0.51), one finds

$$\left[G^{-1}\right]^K = \left[G^R\right]^{-1}\circ F - F\circ\left[G^A\right]^{-1}. \tag{0.54b}$$

The action (0.53) should be viewed as a construction devised to reproduce the proper continuum limit of the correlation functions according to the rules of Gaussian integration. It is fully self–consistent in the following sense: (i) it does not need to appeal to the discrete representation for regularization; (ii) its general structure is intact upon renormalization or “dressing” of its components by the interaction corrections (see chapter **??**).

Here we summarize the main features of the action (0.53), which, for lack of better terminology, we call the *causality structure*:

- The $cl-cl$ component of the quadratic form is zero. It reflects the fact that for a pure classical field configuration ($\phi^q = 0$) the action is zero. Indeed, in this case $\phi^+ = \phi^-$ and the action on the forward part of the contour is canceled by that on the backward part (except for the boundary terms, which are omitted in the continuum limit). The very general statement is, therefore, that

$$S\left[\phi^{cl}, 0\right] = 0\,. \tag{0.55}$$

Obviously this statement is not restricted to the Gaussian action of the form given by Eq. (0.53), but holds for any generic action (see chapter **??**).

- The $cl-q$ and $q-cl$ components are mutually Hermitian conjugated upper and lower (advanced and retarded) triangular matrices in the time domain. This property is responsible for the causality of the response functions as well as for protecting the $cl-cl$ component from a perturbative renormalization (see below). Relations (0.47), (0.48) are crucial for this last purpose and necessary for the consistency of the theory.
- The $q-q$ component is an anti–Hermitian matrix [cf. Eq. (0.45)]. It is responsible for the convergence of the functional integral and keeps information about the distribution function. In our example $\left[G^K\right]^{-1} = 2i0F$, where F is a Hermitian matrix. The fact that it is infinitesimally small is a peculiarity of non-interacting model. We shall see in the following chapters that it receives the finite value, once interactions with other degrees of freedom are included.

0.10 External sources

So far we have been content with the representation of the partition function. The latter does not carry any information in the Keldysh technique, since $Z = 1$. To make the entire construction meaningful one should introduce source fields, which enable one to compute various observables. As an example, let us introduce an external time–dependent potential $V(t)$, defined along the Keldysh contour $\mathcal{C}$. It interacts with the bosons through the Hamiltonian $\hat{H}_V = V(t)\hat{b}^\dagger\hat{b}$. One can now introduce the generating function $Z[V]$ defined similarly to the partition function (0.17) $Z[V] = \mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}[V]\hat{\rho}\}/\mathrm{Tr}\{\hat{\rho}\}$, where the evolution operator $\hat{\mathcal{U}}_{\mathcal{C}}[V]$ includes the source Hamiltonian $\hat{H}_V$ along with the bare one, Eq. (0.16). Repeating the construction of the coherent state functional integral of section 0.4, one obtains for the generating function

$$Z_d[V] = \frac{1}{\mathrm{Tr}\{\hat{\rho}_0\}} \int \prod_{j=1}^{2N} \mathrm{d}[\bar{\phi}_j, \phi_j] \, \exp\left(i \sum_{j,j\,'=1}^{2N} \bar{\phi}_j \, G^{-1}_{jj\,'}[V] \, \phi_{j\,'} \right) , \qquad (0.56)$$

where the subscript d stands for the discrete representation. The $2N \times 2N$ matrix $iG^{-1}_{jj\,'}[V]$ is similar to the one given by Eq. (0.23) with $h_{\mp} \to h_{\mp}[V] = 1 \mp i(\omega_0 + V_j)\delta_t$, where V_j is the value of $V(t)$ at the corresponding discrete time point. According to Eq. (0.24) the generating function is proportional to the inverse determinant of $-iG^{-1}_{jj\,'}[V]$ matrix. The latter is calculated in a way very similar to Eq. (0.28), leading to

$$Z_d[V] = \frac{1}{\mathrm{Tr}\{\hat\rho_0\}}\,\frac{1}{\det\big[-i\hat G^{-1}[V]\big]} = \frac{1-\rho(\omega_0)}{1-\rho(\omega_0)e^{-i\int_{\mathcal{C}}\mathrm{d}t\,V(t)}}\,, \tag{0.57}$$

It is convenient to introduce classical and quantum components of the source potential $V(t)$ as

$$V^{cl}(t) = \frac{1}{2}\left[V^+(t)+V^-(t)\right]\,; \qquad V^q(t) = \frac{1}{2}\left[V^+(t)-V^-(t)\right]\,, \tag{0.58}$$

where $V^{\pm}(t)$ is the source potential on the forward (backward) branch of the contour. With these notations along with Eq. (0.39) the generating function takes the form

$$Z_d[V^{cl},V^q] = \exp\left\{-\ln\left[1-n_B(\omega_0)\left(e^{-2i\int\mathrm{d}t\,V^q(t)}-1\right)\right]\right\}. \tag{0.59}$$

The fact that the generating function depends only on the integral of the quantum component of the source and does not depend on its classical component at all is a peculiarity of our toy model. However, the very general statement is

$$Z[V^{cl},0] = 1\,. \tag{0.60}$$

Indeed, if $V^q = 0$ the source potential is the same on the two branches, $V^+(t) = V^-(t)$, and thus the evolution operator brings the system exactly to its initial state, i.e. $\hat{\mathcal{U}}_{\mathcal{C}}[V^{cl}] = \hat 1$. One crucially needs therefore a fictitious potential $V^q(t)$ to generate observables.

Since the source potential is coupled to the number of particles operator $\hat n = \hat b^\dagger \hat b$, differentiation over $V^q(t)$ generates expectation value of $-2i\langle\hat n(t)\rangle$ (factor of two is due to the fact that we insert $\hat b^\dagger(t)\hat b(t)$ on both branches) $\langle\hat n(t)\rangle = (i/2)\delta Z_d[V^q]/\delta V^q(t)|_{V^q=0} = n_B(\omega_0)$, as was already established in section 0.7. The higher order correlation functions may be obtained by repetitive differentiation of the generating function. To generate *irreducible* correlators $\langle\langle\hat n^k(t)\rangle\rangle \equiv \langle(\hat n(t)-n_B)^k\rangle$ one needs to differentiate the *logarithm* of the generating function, e.g.

$$\begin{aligned}
\langle\langle\hat n^2(t)\rangle\rangle &= \left(\frac{i}{2}\right)^2 \frac{\delta^2\ln Z_d}{\delta[V^q(t)]^2}\bigg|_{V=0} = n_B^2+n_B\,;\\
\langle\langle\hat n^3(t)\rangle\rangle &= \left(\frac{i}{2}\right)^3 \frac{\delta^3\ln Z_d}{\delta[V^q(t)]^3}\bigg|_{V=0} = 2n_B^3+3n_B^2+n_B\,;\\
\langle\langle\hat n^4(t)\rangle\rangle &= \left(\frac{i}{2}\right)^4 \frac{\delta^4\ln Z_d}{\delta[V^q(t)]^4}\bigg|_{V=0} = 6n_B^4+12n_B^3+7n_B^2+n_B\,;
\end{aligned} \tag{0.61}$$

etc.

Let us see now if these results can be reproduced in the continuum technique, without resorting to the discretization. The continuum generating function is defined as

$$Z_c[V] = \int \mathbf{D}[\bar\phi, \phi]\, e^{\,iS[\bar\phi,\phi]+iS_V[\bar\phi,\phi]}\,, \tag{0.62}$$

where the bare action $S[\bar\phi, \phi]$ is given by Eq. (0.34) and

$$S_V[\bar\phi, \phi] = -\int_{\mathcal{C}} \mathrm{d}t\, V(t)\bar\phi(t)\phi(t) = -\int_{-\infty}^{+\infty} \mathrm{d}t \left[V^+\bar\phi^+\phi^+ - V^-\bar\phi^-\phi^-\right] \tag{0.63}$$

$$= -\int_{-\infty}^{+\infty} \mathrm{d}t \left[V^{cl}(\bar\phi^+\phi^+ - \bar\phi^-\phi^-) + V^q\,(\bar\phi^+\phi^+ + \bar\phi^-\phi^-)\right] = -\int_{-\infty}^{+\infty} \mathrm{d}t\, \vec{\bar\phi}^T\hat V\vec\phi,$$

where $\vec\phi = (\phi^{cl}, \phi^q)^T$ and

$$\hat V(t) = \begin{pmatrix} V^q(t) & V^{cl}(t) \\ V^{cl}(t) & V^q(t) \end{pmatrix}. \tag{0.64}$$

As a result, for our example of the single bosonic level the continuum generating function is given by

$$\begin{aligned} Z_c[V^{cl}, V^q] &= \int \mathbf{D}[\bar\phi, \phi]\; e^{\,i\int \mathrm{d}t\, \vec{\bar\phi}^T\left(\hat G^{-1} - \hat V(t)\right)\vec\phi} = \frac{1}{\mathrm{Tr}\{\hat\rho_0\}}\,\frac{1}{\det\left[-i\hat G^{-1} + i\hat V\right]} \\ &= \frac{1}{\det\left[1-\hat G\hat V\right]} = e^{-\mathrm{Tr}\ln\left[1-\hat G\hat V\right]}\,, \end{aligned} \tag{0.65}$$

where we have used Eq. (0.29) along with the identity $\ln\det\hat A = \mathrm{Tr}\ln\hat A$. According to Eqs. (0.43) and (0.49) the matrix Green function is

$$\hat G(t, t') = -ie^{-i\omega_0(t-t')}\begin{pmatrix} f_B(\omega_0) & \theta(t-t') \\ -\theta(t'-t) & 0 \end{pmatrix} \tag{0.66}$$

and $f_B(\omega_0) = 2n_B(\omega_0) + 1$.

The continuum generating function Z_c is *not* identical to the discrete one Z_d. However as we shall show, it possesses the same general properties and generates exactly the same statistics of the number operator. First, let us verify Eq. (0.60) by expanding the logarithm in Eq. (0.65). In the first order in $\hat V$ one finds $-\mathrm{Tr}\ln\left[1-\hat G\hat V\right] \approx \mathrm{Tr}\hat G\hat V = \int \mathrm{d}t\, V^{cl}(t)[G^R(t,t)+G^A(t,t)] = 0$, where we put $V^q = 0$ and employed Eq. (0.47). In the second order one encounters $\int \mathrm{d}t\mathrm{d}t' V^{cl}(t)G^R(t,t')V^{cl}(t')G^R(t',t)$ and similarly for G^A. Since $G^R(t,t') = 0$ if $t < t'$, while $G^R(t',t) = 0$ if $t > t'$, the expression under the integral is non-zero only if $t = t'$. In the continuum limit ($N \to \infty$) this is the manifold of zero measure, making the integral to be zero. Clearly the same holds in all orders in V^{cl}. This illustrates how the generic feature of the Keldysh technique, Eq. (0.60), works in our simple example.

Consider now $i\delta Z_c[V]/\delta V^q(t)|_{V=0} = \langle \bar{\phi}^+(t)\phi^+(t) + \bar{\phi}^-(t)\phi^-(t)\rangle$, we refer to Eqs. (0.62) and (0.63) to see this relation. Expectation value of which operator is calculated this way? The naive answer is that $\bar{\phi}(t)\phi(t)$ is generated by $\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle$ and we deal with the sum of this operator inserted on the forward and backward branches. If this would be the case, $\bar{\phi}$ would be taken one time step ahead of ϕ field, as is indeed the case in the discrete representation. However, our continuum expression indiscriminately places both $\bar{\phi}^\pm$ and $\phi^\pm$ at the same time t. One can check that such a "democratic" choice of the time arguments corresponds to the expectation value of the symmetric combination $\hat{f}(t) \equiv \hat{b}^\dagger(t)\hat{b}(t) + \hat{b}(t)\hat{b}^\dagger(t)$. Employing the equal time commutation relation $[\hat{b}(t), \hat{b}^\dagger(t)] = \hat{1}$, one finds $\hat{f}(t) = 2\hat{n}(t) + 1$ and $\langle \hat{f}(t)\rangle = i\,\delta Z_c[V^{cl}, V^q]/\delta V^q(t)\big|_{V=0} = iG^K(t,t) = f_B(\omega_0)$ as it should be, of course. For higher order irreducible correlators one obtains

$$
\begin{aligned}
\langle\langle \hat{f}^2(t)\rangle\rangle &= i^2\, \frac{\delta^2 \ln Z_c}{\delta[V^q(t)]^2}\bigg|_{V=0} = f_B^2 - 1\,; \\
\langle\langle \hat{f}^3(t)\rangle\rangle &= i^3\, \frac{\delta^3 \ln Z_c}{\delta[V^q(t)]^3}\bigg|_{V=0} = 2f_B^3 - 2f_B\,; \\
\langle\langle \hat{f}^4(t)\rangle\rangle &= i^4\, \frac{\delta^4 \ln Z_c}{\delta[V^q(t)]^4}\bigg|_{V=0} = 6f_B^4 - 8f_B^2 + 2\,;
\end{aligned}
\tag{0.67}
$$

etc. To see how it works, consider e.g. the third order term in the expansion of the logarithm in Eq. (0.65) in powers of $V^q(t)$ at $V^{cl} = 0$

$$
\begin{aligned}
&\frac{1}{3}\mathrm{Tr}\{(\hat{G}\hat{V})^3\} = \frac{1}{3}\int \mathrm{d}t\mathrm{d}t'\mathrm{d}t''\,\mathrm{Tr}\left\{\hat{G}(t,t')V^q(t')\hat{G}(t',t'')V^q(t'')\hat{G}(t'',t)V^q(t)\right\} \\
&= i\frac{f_B^3}{3}\left(\int \mathrm{d}tV^q(t)\right)^3 - if_B\int \mathrm{d}tV(t)\left(\int_t \mathrm{d}t'V^q(t')\right)^2 = i\frac{f_B^3 - f_B}{3}\left(\int \mathrm{d}tV^q(t)\right)^3 .
\end{aligned}
$$

To calculate the last integral in the intermediate expression here one introduces $W(t) = \int_t V^q(t)$ and therefore $V^q = -\dot{W}$, the integral in question is thus $-\int \mathrm{d}t\dot{W}W^2 = -\int \mathrm{d}WW^2 = -(1/3)W^3(t)|_{-\infty}^{\infty} = (1/3)(\int \mathrm{d}tV^q)^3$. Differentiating over V^q three times, one arrives at Eq. (0.67).

Substituting $\hat{f} = 2\hat{n} + 1$ and $f_B = 2n_B + 1$, it is easy to check that the respective momemnts (0.61) and (0.67) are exactly equivalent! Therefore, although the generating functions Z_d and Z_c generate slightly different set of correlators, their statistical content is equivalent. From now on we shall always deal with the continuum version, circumventing the tedious discretization procedure.

The generating function $Z[V^q]$ gives an access not only to the moments, but to a *full quantum statistics* of the operator $\hat{n}(t_0)$, or $\hat{f}(t_0)$. Let us define the probability to measure n bosons at a time t_0 as $P(n)$. Then $\langle \hat{n}^k(t_0)\rangle = \int \mathrm{d}n\, n^k P(n)$. The generating function $Z[\lambda] \equiv \int \mathrm{d}n\, e^{i\lambda n} P(n) = \sum_k (i\lambda)^k \langle \hat{n}^k(t_0)\rangle / k!$, where λ is called counting "field". Comparing this with $Z_d[V^q]$, one notices that $Z[\lambda]$ may be obtained by the substitution $V^q(t) = -(\lambda/2)\delta(t-t_0)$. Employing Eq. (0.57), one finds

$$Z[\lambda] = \frac{1-\rho(\omega_0)}{1-\rho(\omega_0)\, e^{i\lambda}} = \big(1-\rho(\omega_0)\big) \sum_{k=0}^{\infty} [\rho(\omega_0)]^k \, e^{ik\lambda}\,. \tag{0.68}$$

Performing the inverse Fourier transform and recalling that $\rho(\omega_0) = e^{-\beta(\omega_0-\mu)}$, one finds

$$P(n) = \sum_{k=0}^{\infty} \delta(n-k) \left(1 - e^{-\beta(\omega_0-\mu)}\right) e^{-\beta(\omega_0-\mu)k}\,. \tag{0.69}$$

I.e. one can measure only integer number of bosons and the corresponding probability is proportional to $e^{-\beta(E_n-\mu n)}$, where the energy $E_n = n\omega_0$. This is, of course trivial result, which we have already de-facto employed in Eq. (0.18). The important message, however, is that the counting field λ is nothing but a particular realization of the quantum source field $V^q(t)$, tailored to generate an appropriate statistics. As opposed to the calculation of the moments (0.61), (0.67), one should *not* put the quantum source to zero when the *full statistics* is evaluated. We shall employ this lesson in chapters **?? ??**, when we shall discuss much less obvious examples of the full quantum statistics.

0.11 Harmonic oscillator

The simplest many–body system of a single bosonic state, considered above, is, of course, equivalent to a single–particle harmonic oscillator. To make this connection explicit, consider the Keldysh contour action Eq. (0.30) with the correlator Eq. (0.33) written in terms of the complex field $\phi(t)$. The latter may be parameterized by its real and imaginary parts as

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}} \big(P(t) - i\,\omega_0\, X(t)\big)\,, \qquad \bar{\phi}(t) = \frac{1}{\sqrt{2\omega_0}} \big(P(t) + i\,\omega_0\, X(t)\big)\,. \tag{0.70}$$

In terms of the real fields $P(t)$ and $X(t)$ the action, Eq. (0.30), takes the form

$$S[X,P] = \int_{\mathcal{C}} \mathrm{d}t \left[P\,\dot{X} - \frac{1}{2}\,P^2 - \frac{\omega_0^2}{2}\,X^2 \right] , \tag{0.71}$$

where the full time derivatives of P^2, X^2 and $P\,X$ were omitted, since they contribute only to the boundary terms, not written explicitly in the continuum notations (they have to be kept for the proper regularization, though). Equation (0.71) is nothing but the action of the quantum harmonic oscillator in the Hamiltonian form. One may perform the Gaussian integration over the real field $P(t)$ to obtain

$$S[X] = \int_{\mathcal{C}} \mathrm{d}t \left[\frac{1}{2}\,\dot{X}^2 - \frac{\omega_0^2}{2}\,X^2 \right] . \tag{0.72}$$

This is the Feynman Lagrangian action of the harmonic oscillator, written on the Keldysh contour. It may be generalized for an arbitrary single particle potential $U(X)$

$$S[X] = \int_{\mathcal{C}} \mathrm{d}t \left[\frac{1}{2}\,\dot{X}^2 - U(X) \right] . \tag{0.73}$$

One may split the $X(t)$ field into two components: $X^+(t)$ and $X^-(t)$, residing on the forward and backward branches of the contour. The Keldysh rotation for real fields is convenient to define as

$$X^{cl}(t) = \frac{1}{2}\left[X^+(t) + X^-(t)\right] ; \qquad X^{q}(t) = \frac{1}{2}\left[X^+(t) - X^-(t)\right] . \tag{0.74}$$

In terms of these fields the action takes the form

$$S[X^{cl}, X^q] = \int_{-\infty}^{+\infty} \mathrm{d}t \left[-2X^q\ddot{X}^{cl} - U\big(X^{cl} + X^q\big) + U\big(X^{cl} - X^q\big) \right] , \tag{0.75}$$

where the integration by parts was performed in the term $\dot{X}^q\dot{X}^{cl}$. This is the Keldysh form of the Feynman path integral. The omitted boundary terms provide a convergence factor of the form $\sim i0(X^q)^2$.

If the fluctuations of the quantum component $X^q(t)$ are regarded as small, one may expand the potential to the first order and find for the action

$$S[X^{cl}, X^q] = -\int_{-\infty}^{+\infty} \mathrm{d}t \left[2X^q \left(\ddot{X}^{cl} + U'(X^{cl}) \right) + O\big[(X^q)^3\big] \right] , \tag{0.76}$$

where $U'(X) = \partial U(X)/\partial X$. In this approximation the integration over the quantum component, X^q, may be explicitly performed, leading to the functional delta-function of the expression in the round brackets. This delta-function enforces the classical Newtonian dynamics of X^{cl}

$$\ddot{X}^{cl} = -U'\big(X^{cl}\big)\,. \tag{0.77}$$

This is the reason the symmetric, over forward and backward branches, part of the Keldysh field is called the classical component. One should be careful with this name, though. If the higher order terms in X^q are kept in the action, both X^q and X^{cl} are subject to quantum fluctuations.

Returning back to the harmonic oscillator, $U(X) = \omega_0^2 X^2/2$, one may rewrite its Feynman-Keldysh action in the matrix form

$$S[\vec{X}] = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t\, \vec{X}^T \hat{D}^{-1} \vec{X}\,, \tag{0.78}$$

where in analogy with the complex field, Eq. (0.53), we introduced

$$\vec{X}(t) = \left(\begin{array}{c} X^{cl}(t) \\ X^q(t) \end{array}\right)\,; \qquad \hat{D}^{-1} = \left(\begin{array}{cc} 0 & [D^{-1}]^A \\ [D^{-1}]^R & [D^{-1}]^K \end{array}\right) \tag{0.79}$$

and superscript T stands for matrix transposition. Here the retarded and advanced components of the quadratic form in the action are given by $\frac{1}{2}[D^{-1}]^{R(A)} = (i\partial_t \pm i0)^2 - \omega_0^2$. As before, one should understand that this expression is simply a continuous abbreviation for the large lower (upper) triangular matrices with $-\delta_t^{-1}$ along the main diagonal, $2\delta_t^{-1} - \omega_0^2\delta_t$ along the lower (upper) sub-diagonal and $-\delta_t^{-1}$ along the second lower (upper) sub-diagonal. This makes $\hat{D}^{-1}$ matrix symmetric, since its $[D^{-1}]^K$ component must be symmetric by construction (its antisymmetric part does not enter the action). In continuous notations the Keldysh component $[D^{-1}]^K$ is only a regularization. It is convenient to keep it explicitly, since it suggests the way the matrix $\hat{D}^{-1}$ should be inverted to find the correlation functions

$$\Big\langle X^\alpha(t) X^\beta(t')\Big\rangle = \int \mathbf{D}[\vec{X}]\, X^\alpha(t) X^\beta(t')\, e^{\,iS[\vec{X}]} = i\hat{D}^{\alpha\beta}(t,t')\,, \tag{0.80}$$

where $\alpha, \beta = (cl, q)$ and the correlation matrix is given by

$$\hat{D}^{\alpha\beta}(t,t') = \left(\begin{array}{cc} D^K(t,t') & D^R(t,t') \\ D^A(t,t') & 0 \end{array}\right)\,. \tag{0.81}$$

To apply the rules of the Gaussian integration for real variables (see section 0.5), it is crucial that the matrix $\hat{D}^{-1}$ is symmetric. In the Fourier representation components of the correlation matrix are given by

$$D^{R(A)}(\epsilon) = \frac{1}{2}\,\frac{1}{(\epsilon \pm i0)^2 - \omega_0^2}\,, \tag{0.82a}$$

$$D^K(\epsilon) = \coth\frac{\epsilon}{2T}\left[D^R(\epsilon) - D^A(\epsilon)\right] , \qquad (0.82b)$$

where we have assumed an equilibrium thermal distribution with zero chemical potential. One way to check the consistency of the expression for the Keldysh component is to express X^α through $\bar\phi^\alpha$ and ϕ^α and employ the correlation functions for the complex fields, derived in Chapter **??**.

The normalization identity $\int \mathbf{D}[\vec{X}]\, e^{iS[\vec{X}]} = 1$, is maintained in the following way: (i) first, due to the structure of $\hat{D}^{-1}$ matrix, explained above, $\det[\frac{1}{i}\hat{D}^{-1}] = -\det[\frac{1}{i}D^{-1}]^R \det[\frac{1}{i}D^{-1}]^A = (2/\delta_t)^{2N}$; (ii) the integration measure is understood as $\mathbf{D}[\vec{X}] = \prod_{j=1}^N 2\left(\mathrm{d}X_j^{cl}/\sqrt{2\pi\delta_t}\right)\left(\mathrm{d}X_j^q/\sqrt{2\pi\delta_t}\right)$ (in comparison with Eq. (0.26) there is an additional factor of 2, which originates from the Jacobian of the transformation (0.74), and factor δ_t^{-1} at each time slice, coming from the integration over $P(t)$). According to Eq. (0.26) this leads exactly to the proper normalization. One can also understand the normalization in a way discussed below Eq. (0.54a), without resorting to the discrete representation.

0.12 Quantum particle in contact with an environment

Consider a quantum particle with the coordinate $X(t)$, placed in a potential $U(X)$ and brought into a contact with a bath of harmonic oscillators. The bath oscillators are labeled by an index s and their coordinates are denoted as φ_s. They posses a set of frequencies ω_s. The Keldysh action of such a system is given by the three terms $S = S_\mathrm{p} + S_\mathrm{bath} + S_\mathrm{int}$, where

$$S_\mathrm{p}[X] = \int_{-\infty}^{+\infty}\mathrm{d}t\left[-2X^q\ddot{X}^{cl} - U\left(X^{cl}+X^q\right) + U\left(X^{cl}-X^q\right)\right], \qquad (0.83a)$$

$$S_\mathrm{bath}[\varphi_s] = \frac{1}{2}\sum_s\int_{-\infty}^{+\infty}\mathrm{d}t\,\vec{\varphi}_s^{\,T}\hat{D}_s^{-1}\vec{\varphi}_s\,, \qquad (0.83b)$$

$$S_\mathrm{int}[X,\varphi_s] = \sum_s g_s\int_{-\infty}^{+\infty}\mathrm{d}t\,\vec{X}^T\,\hat{\sigma}_1\,\vec{\varphi}_s\,, \qquad (0.83c)$$

where the symmetric quadratic form $\hat{D}_s^{-1}$ is given by Eq. (0.79) with the frequency ω_s. The interaction term between the particle and the bath oscillators is taken as a product of their coordinates $\sum_s g_s\int_\mathcal{C}\mathrm{d}t\,X(t)\varphi_s(t) = \sum_s g_s\int\mathrm{d}t(X^+\varphi_s^+ - X^-\varphi_s^-)$. Performing the Keldysh rotation according to Eq. (0.74), one arrives at Eq. (0.83c), where $\hat{\sigma}_1$ is the first Pauli matrix. The corresponding coupling constants are denoted as g_s.

One may now integrate out the degrees of freedom of the bath to reduce the problem to the particle coordinate only. Employing Eq. (0.26) for the Gaussian integration over the real variables, one arrives at the so–called dissipative action for the particle

$$S_{\rm diss} = \frac{1}{2}\iint_{-\infty}^{+\infty} {\rm d}t\,{\rm d}t'\,\vec{X}^{\,T}(t)\,\hat{\mathfrak{D}}^{-1}(t-t')\,\hat{X}(t')\,, \qquad (0.84a)$$

$$\hat{\mathfrak{D}}^{-1}(t-t') = -\hat{\sigma}_1\sum_s g_s^2 \hat{D}_s(t-t')\,\hat{\sigma}_1\,. \qquad (0.84b)$$

The straightforward matrix multiplication shows that the dissipative quadratic form $\hat{\mathfrak{D}}^{-1}$ possesses the causality structure as e.g. Eq. (0.79). For the Fourier transform of its retarded (advanced) components, one finds:

$$\left[\mathfrak{D}^{-1}(\epsilon)\right]^{R(A)} = -\frac{1}{2}\sum_s \frac{g_s^2}{(\epsilon\pm i0)^2-\omega_s^2} = \int\limits_0^\infty \frac{{\rm d}\omega}{2\pi}\,\frac{\omega J(\omega)}{\omega^2-(\epsilon\pm i0)^2}\,, \qquad (0.85)$$

where $J(\omega)=\pi\sum_s(g_s^2/\omega_s)\delta(\omega-\omega_s)$ is the bath spectral density.

We shall assume now that the spectral density behaves as $J(\omega)=8\gamma\omega$, where γ is a constant at small frequencies. This is the so-called *ohmic* bath, which is frequently found in more realistic models of the environment. Substituting it into Eq. (0.85), one finds

$$\left[\mathfrak{D}^{-1}(\epsilon)\right]^{R(A)} = 4\gamma\int\frac{{\rm d}\omega}{2\pi}\,\frac{\omega^2}{\omega^2-(\epsilon\pm i0)^2} = {\rm const}\pm 2i\gamma\epsilon\,, \qquad (0.86)$$

where ϵ–independent real constant (same for R and A components) may be absorbed into the redefinition of the harmonic part of the potential $U(X)=$ const $X^2+\ldots$ and, thus, may be omitted. If the bath is in equilibrium the Keldysh component of the correlator is set by FDT

$$\left[\mathfrak{D}^{-1}(\epsilon)\right]^{K} = \left(\left[\mathfrak{D}^R\right]^{-1}-\left[\mathfrak{D}^A\right]^{-1}\right)\coth\frac{\epsilon}{2T} = 4i\gamma\epsilon\,\coth\frac{\epsilon}{2T}\,, \qquad (0.87)$$

where we assumed that the bath is at temperature T and put the chemical potential of the bath excitations to be zero. Notice that the validity of this expression does *not* rely on the particle particle being at equilibrium, but only the bath. The Keldysh component is an anti–Hermitian operator with a positive–definite imaginary part, rendering convergence of the functional integral over $\vec{X}(t)$.

In the time representation the retarded (advanced) component of the correlator takes a time–local form: $\left[\mathfrak{D}^{R(A)}\right]^{-1}=\mp 2\gamma\,\delta(t-t')\,\partial_{t'}$. On the other hand, the Keldysh component is a non–local function, that may be found by the inverse Fourier transform of Eq. (0.87):

$$\left[\mathfrak{D}^{-1}(t-t')\right]^K = 4i\gamma\left[(2T+C)\delta(t-t') - \frac{\pi T^2}{\sinh^2[\pi T(t-t')]}\right], \qquad (0.88)$$

where the infinite constant C serves to satisfy the condition $\int dt[\mathfrak{D}^{-1}(t)]^K = [\mathfrak{D}^{-1}(\epsilon=0)]^K = 8i\gamma T$. Finally, one obtains for the Keldysh action of the particle connected to the ohmic bath

$$S[\vec{X}] = \int_{-\infty}^{+\infty} \mathrm{d}t\left[-2X^q\left(\ddot{X}^{cl} + \gamma\dot{X}^{cl}\right) - U\left(X^{cl} + X^q\right) + U(X^{cl} - X^q)\right]$$
$$+\,2i\gamma\int_{-\infty}^{+\infty} \mathrm{d}t\left[2T\big(X^q(t)\big)^2 + \frac{\pi T^2}{2}\int_{-\infty}^{+\infty} \mathrm{d}t'\,\frac{\big(X^q(t) - X^q(t')\big)^2}{\sinh^2[\pi T(t-t')]}\right]. \quad (0.89)$$

This action satisfies all the causality criteria listed in Sec. 0.9. Notice, that in the present case the Keldysh ($q-q$) component is not just a regularization, but a finite term, originating from the coupling to the bath and serving to limit fluctuations. The other manifestation of the bath is the presence of the friction term, $\sim \gamma\partial_t$ in the R and the A components. In equilibrium the friction coefficient and fluctuations amplitude are rigidly connected by the FDT. The quantum dissipative action, Eq. (0.89), is a convenient playground to demonstrate various approximations and connections to other approaches. We shall discuss it in details in chapter **??**. If only linear terms in X^q are kept in the action (0.89), the integration over $X^q(t)$ results in the functional delta function, which enforces the following relation

$$\ddot{X}^{cl} = -U'\big(X^{cl}\big) - \gamma\dot{X}^{cl}\,. \qquad (0.90)$$

This is the classical Newtonian equation with the viscous friction force. Remarkably, we have obtained $\dot{X}^{cl}$ term in the equation of motion from the action principle. It would not be possible, if not for the doubling of the number of fields X^{cl} and X^q. Indeed, in any action, depending on X^{cl} only, terms linear in the first time derivative may be integrated out and thus do not affect the equation of motion.

0.13 From Matsubara to Keldysh

Most of the texts, dealing with equilibrium systems at finite temperature, employ Matsubra technique [18, 19]. This method is designed to treat the equilibrium density matrix $e^{-\beta\hat{H}}$ as the evolution operator. To this end one considers an imaginary time quantum mechanics, with imaginary time τ restricted to the interval $0 \le \tau < \beta$. When calculating an expectation value of an observable $\hat{\mathcal{O}}(\tau)$, one evaluates trace of the form $\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}\{\hat{\mathcal{O}}(\tau)e^{-\beta\hat{H}}\}$. To this end one divides the imaginary time interval $[0,\beta]$ onto N infinitesimal segments and inserts the resolution of unity in the coherent state basis at each segment, similarly to our procedure in section 0.4. As a result, one ends up with the fields, say coordinate $X(\tau)$, which in view of fact that one evaluates the trace obeys the periodic boundary conditions, $X(0) = X(\beta)$. In the Fourier representation it is represented by a discrete set of components $X_m = \int_0^\beta \mathrm{d}\tau X(\tau)\, e^{i\epsilon_m\tau}$, where $\epsilon_m = 2\pi mT$ is a set of Matsubra frequencies and m is an integer.

We shall discuss now, how to convert an action written in the Matsubara technique into the Keldysh representation. This may be useful, if one wishes to extent treatment of the problem to non-equilibrium or time-dependent conditions. As an example consider the following bosonic Matsubara action:

$$S[X_m] = \frac{i}{2}\,\gamma T \sum_{m=-\infty}^{\infty} |\epsilon_m||X_m|^2\,, \tag{0.91}$$

Due to the absolute value sign: $|\epsilon_m| \neq i\partial_\tau$. In fact, in the imaginary time representation the kernel $F_m = |\epsilon_m|$ acquires the form $F(\tau) = \sum_m |\epsilon_m| e^{-i\epsilon_m\tau}$ $= C\delta(\tau) - \pi T\sin^{-2}(\pi T\tau)$, where the infinite constant C is chosen to satisfy $\int_0^\beta \mathrm{d}\tau F(\tau) = F_0 = 0$. As a result, in the imaginary time representation the action (0.91) obtains the following non-local form

$$\begin{aligned} S[X] &= \frac{i}{2}\,\gamma T \iint_0^\beta \mathrm{d}\tau\, \mathrm{d}\tau'\, X(\tau)\, F(\tau-\tau')\, X(\tau') \\ &= \frac{i}{4\pi}\,\gamma \iint_0^\beta \mathrm{d}\tau\, \mathrm{d}\tau'\, \frac{\pi^2 T^2}{\sin^2[\pi T(\tau-\tau')]} \big(X(\tau) - X(\tau')\big)^2 . \end{aligned} \tag{0.92}$$

This action is frequently named after Caldeira and Leggett [?], who used it to investigate the influence of dissipation on quantum tunneling.

To transform to the Keldysh representation one proceeds along the following steps: (i) double the number of degrees of freedom, correspondingly doubling the action, $X \to \vec{X} = (X^{cl}, X^q)^T$ and consider the latter as functions of the real time t or real frequency ϵ; (ii) according to the causality structure, Sec. 0.9, the general form of the quadratic, time translationally invariant Keldysh action is:

$$S[\vec{X}] = \gamma \int \frac{\mathrm{d}\epsilon}{2\pi}\, (X^{cl}_\epsilon, X^q_\epsilon) \begin{pmatrix} 0 & F^A(\epsilon) \\ F^R(\epsilon) & F^K(\epsilon) \end{pmatrix} \begin{pmatrix} X^{cl}_\epsilon \\ X^q_\epsilon \end{pmatrix}; \tag{0.93}$$

(iii) the retarded (advanced) component $F^{R(A)}(\epsilon)$ is the analytic continuation of the Matsubara correlator $F(\epsilon_m) = |\epsilon_m|$ from the *upper (lower)* half–

plane of the complex variable ϵ_m to the real axis: $-i\epsilon_m \to \epsilon$, see Ref. [19]. As a result, $F^{R(A)}(\epsilon) = \pm i\epsilon$; (iv) in equilibrium the Keldysh component follows from FDT: $F^K(\epsilon) = \big(F^R(\epsilon) - F^A(\epsilon)\big)\coth(\epsilon/2T) = 2i\epsilon \coth(\epsilon/2T)$, cf. Eqs. (0.86) and (0.87). We found thus that $\gamma\hat{F}(\epsilon) = \frac{1}{2}\hat{\mathfrak{D}}^{-1}(\epsilon)$ and therefore the Keldysh counterpart of the Matsubara action, Eqs. (0.91) or (0.92) is the already familiar dissipative action (0.89), (without the potential and inertial terms, of course). One may now include external fields and allow the system to deviate from the equilibrium.

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